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Inflation, Exchange Rates and PPP in a Multivariate Panel Cointegration Model*

Tor Jacobson, Johan Lyhagen†, Rolf Larsson and Marianne Nessén

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Abstract

New multivariate panel cointegration methods are used to analyze nominal exchange rates and prices in four major economies in Europe; France, Germany, Italy and the United Kingdom for the post-Bretton Woods period. We test for purchasing power parity between these four countries and find that the theoretical PPP relationship does not hold. However, the estimated unrestricted relationship is found to be remarkably close to the theoretical one (1,-1.5,0.9 instead of 1,-1,1). Relevant asymptotic results are stated, proved, and evaluated using Monte Carlo simulations. The asymptotic results are general and may hence be used in similar empirical contexts using the same model structure. Parametric bootstrap inference is used in order to deal with test size distortions.

Key words: Panel data, long-run purchasing power parity, multivariate cointegration analysis, bootstrap inference.

JEL Classification: F30, C15, C32.

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1 Introduction

Does purchasing power parity (PPP) hold in the long run? Are real exchange rates mean-reverting? Evidence of long-run PPP is often found if one applies unit-root tests to real exchange rate data spanning long periods of time (say, close to a century or more), see, e.g., Frankel 1986, Abuaf and Jorion 1990 and Lothian and Taylor 1996. However, when examining the recent post-Bretton Woods period of floating exchange rates conventional unit-root tests do not, in general, find evidence of PPP. Nor do studies using the multivariate maximum likelihood cointegration method to analyze long-run PPP using post-Bretton Woods data. Cheung and Lai (1993), Kugler and Lenz (1993), Johansen and Juselius (1992), MacDonald (1993) and Edison, Gagnon and Melick (1997) examined PPP in trivariate systems with a nominal exchange rate and two price indices, whereas Nessén (1996) estimated a larger model for three countries. The typical result in these studies is that evidence of cointegration is found, but that the cointegrating relations fail to comply with the restrictions implied by PPP.

The inadequate power associated with pure time series testing for unit roots and cointegration subsequently guided PPP-research towards panel data applications. Unit roots in real exchange rates have been examined in panels of post-Bretton Woods data by, i.a., Frankel and Rose (1996), O'Connell (1998), Oh (1996), Papell (1997) and Wu (1996), again with mixed results. Following Pedroni (1995, 1996, and 1997) a number of panel data applications test for cointegration between nominal exchange rates and prices; Chinn (1997), Obstfeld and Taylor (1996), and Taylor (1996). As in the case of multivariate time series applications the results suggest cointegration between these variables, but not according to PPP.

The purpose of this paper is to re-examine the issue of PPP. Our analysis is based on the recent approach to testing for cointegration in heterogenous panel data models suggested by Larsson, Lyhagen, and Löthgren (2001). Their approach extends the likelihood inference for cointegrated vector autoregressive models developed by Johansen (1988, 1991, 1995) into a panel data setting. This means that the model benefits from the generality and flexibility of maximum likelihood cointegration analysis, as well as the advantage of a vastly enlarged information set offered by a panel data approach. However, the model in Larsson et al. (2001) is not immediately applicable for testing hypotheses about PPP; for that purpose it is too restrictive. The Larsson et al. (2001) model assumes complete independence between the panels as reflected by block diagonal long-run, short-run, and covariance matrices. Groen and Kleibergen (1999) relax the assumption of a block diagonal covariance matrix. In a subsequent paper, Larsson and Lyhagen (1999), all block diagonal restrictions are relaxed except for the matrix containing the cointegrating vectors. However, the interdependent nature of the foreign exchange markets suggest that a block diagonal cointegration matrix is inappropriate, i.e., the PPP hypothesis implies that long-run relations between prices and exchange rates in different countries are indeed related. Lyhagen (2000) shows that discarding this dependence in a standard panel unit root test in the context of PPP gives rise to invalid inference, i.e., the size of the test tends to unity as the number of countries increases. This is due to the presence of a commonly shared common trend that is not taken into account of when calculating the critical values. The Larsson and Lyhagen (1999) model needs to be adjusted in order to reflect the PPP-dependence and, as a consequence, new results for the limit distributions of the test statistics have to be derived.

The empirical analysis in this paper is carried out with one important issue in mind, namely the problem of test size distortion, i.e., the erroneous rejection of a true null hypothesis due to inappropriate critical values. There is reason to believe that the usefulness of multivariate maximum likelihood cointegration analysis can be severely hampered by the curse of dimensionality arising from a large number of parameters in relation to a small number of observations. One undesirable effect is that asymptotic distributions provide poor approximations in small-sample applications and yield inference plagued by size distortions. This has been empirically verified in Jacobson, Vredin and Warne (1998). Gredenhoff and Jacobson (2001) have confirmed the presence and examined the nature of size distortion for likelihood ratio tests of linear restrictions on cointegrating vectors. In this paper, asymptotic tests are augmented with parametric bootstrap analogues, whereby we reduce, if not eliminate, the size distortions due to the small-sample effect.

We examine monthly data for the post-Bretton Woods years 1974-1999 for France, Germany, Italy and the United Kingdom, and the results suggest the following. We do find evidence of cointegration between nominal exchange rates and prices; in fact the number of cointegrating vectors is precisely what PPP predicts, namely one. Moreover, we cannot reject the hypothesis that the panel individual cointegrating vectors are identical. But the final test for PPP fails in that the coefficients in the cointegrating vectors are not compatible with the theoretical PPP-relationship. However, although we reject PPP, it is interesting to note that the estimated unrestricted relationship is found to be remarkably close to the theoretical one, with coefficient estimates (1, -1.5, 0.9) compared with (1, -1, 1). This can be interpreted as support for the view that purchasing power parity is, after all, a reasonable approximation.

We discuss these results in the concluding Section 6. Prior to that, Section 2 explains how a multivariate cointegration panel data model can be formulated for investigating the existence of PPP. Section 3 presents asymptotic results for the hypothesis tests in the statistical model, whereas proofs of the theorems are spelled out in the Appendix. Section 4 contains the empirical analysis. In Section 5 we undertake Monte Carlo simulations in order to disclose the small-sample properties of the proposed asymptotic tests.

2 PPP and linear restrictions on prices and exchange rates

We examine long-run PPP between four large European economies in a multivariate panel setting. The purpose of this section is to show how such a system can be set up and to identify the restrictions implied by long-run PPP.

Denote the natural logarithm of the nominal British pound exchange rate of country i (that is, the number of currency i per unit British pound) by e_t^i . Further, let p_t^i be the natural logarithm of the price level in country i. Further, let p_t^* denote the price level in our numeraire country, the United Kingdom. Define

$$X_{it} = \begin{bmatrix} e_t^i \\ p_t^i \end{bmatrix}$$

and then

$$X_t = \begin{bmatrix} e_t^1 \\ p_t^1 \\ \vdots \\ e_t^N \\ p_t^N \\ p_t^* \end{bmatrix}$$

where N is the number of countries except the base country, in our case three.

Now, if long-run bilateral PPP holds then the real exchange rates between all pairs of countries are stationary, or integrated of order 0, $I(0)$. This may be expressed as

$$q_t^i \equiv e_t^i - p_t^i + p_t^* \sim I(0) \qquad\qquad i = 1, ...N$$

where q_t^i is the real exchange rate between country i and the United Kingdom. These N equations can be summarized as:

$$\begin{bmatrix} q_t^1 \\ q_t^2 \\ \\ \vdots \\ \\ q_t^N \end{bmatrix} \equiv \begin{bmatrix} 1 & -1 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & \dots & 0 & 0 & 1 \\ \vdots & \vdots & & & \ddots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} e_t^1 \\ p_t^1 \\ \vdots \\ e_t^N \\ p_t^N \\ p_t^* \end{bmatrix} \sim I(0) \tag{1}$$

It is easily recognized that the choice of base country is arbitrary. Pre-multiplying the relationship with the matrix

$$\begin{bmatrix} 1 & 0 & \cdots & -1 & \cdots & 0 \\ 0 & 1 & & -1 & \cdots & 0 \\ \vdots & & \ddots & \vdots & & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 0 \\ \vdots & & & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & \cdots & 1 \end{bmatrix}$$

where the column of -1 is in the position of the new base country, gives the desired result. Note that the eigenvalues are $N-1$ ones and the last is minus one so the new relationships span the same space as the original one.

The equations in (1) can be evaluated in a vector error correction model on the form

$$\Delta X_t = \alpha\beta' X_{t-1} + \sum_{j=1}^{m-1} \Gamma_j \Delta X_{t-j} + \mu + \varepsilon_t, \tag{2}$$

where $\alpha'_\perp \mu \neq 0$, with $\alpha_\perp$ such that $\alpha'_\perp \alpha = 0$ and $(\alpha, \alpha_\perp)$ has full rank. (This means that μ is not restricted to the cointegration space.) Moreover, α and β are $N_p \times Nr$, where $N_p \equiv Np + 1$ and β is given by

$$\beta = \begin{bmatrix} \beta_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \dots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{bmatrix}, \tag{3}$$

where for $i = 1, ..., N$, the β_i are $p \times r$ and the $\beta_{N+1,i}$ are $1 \times r$. No restrictions are imposed on the α, Γ_j $(N_p \times N_p)$ and Ω $(N_p \times N_p)$ matrices, the latter being the covariance matrix of ε_t $(N_p \times 1)$. Assume that observations are taken at $t = 1, ..., T$. Note that this model is similar to the one in Larsson and Lyhagen (1999) with the addition of the last row in the β matrix, and the estimation procedures follow those outlined there.

If PPP holds, we have $p = 2$, $r = 1$ and $\left[\beta_i', \beta_{N+1,i}'\right] = [1, -1, 1]$. This is a restriction in model (2) that we will test in three steps. First, we will estimate r, using the sequential testing procedure due to Johansen (1995), i.e., first test if $r = 0$ against $r = p$, then if the null hypothesis is rejected test if $r = 1$, and so on, until the null hypothesis cannot be rejected. This null hypothesis then gives us the estimated r. If the estimated r turns out to be 1, we will go on and test if all $\left[\beta_i', \beta_{N+1,i}'\right]'$ span the same (cointegration) space. Finally, if the hypothesis of a common cointegration space is not rejected, we will test that all $\left[\beta_i', \beta_{N+1,i}'\right] = c_i\,[1, -1, 1]$, where the c_i are constants.

The limit distributions of these tests are considered in the next section. The asymptotic results are general, and can be used in other contexts with the same model structure.

3 Asymptotic results

The hypotheses to be discussed in this section are

$\mathcal{H}_4$: $\operatorname{rank}(\Pi) \leq Np + 1$,

$\mathcal{H}_3$: $\Pi = \alpha\beta'$ where α and β are $(Np + 1) \times Nr$ as above, but with no restrictions on β,

$\mathcal{H}_2$: as $\mathcal{H}_3$ but where β is as in (3),

$\mathcal{H}_1$: as $\mathcal{H}_2$ but where all $\left(\beta_i', \beta_{N+1,i}'\right)'$ span the same space,

$\mathcal{H}_0$: as $\mathcal{H}_1$ but where all $\left(\beta_i', \beta_{N+1,i}'\right)' = H_i\psi_i$, where the H_i are known $(p + 1) \times s$ matrices and the ψ_i are $s \times r$, unknown and span the same space. Obviously, $\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \mathcal{H}_3 \subset \mathcal{H}_4$. For $i < j$, we will denote the maximum likelihood ratio between $\mathcal{H}_i$ and $\mathcal{H}_j$ by Q_{ij}. The theorems will give the asymptotic distributions of, in turn, $-2\log Q_{24}$, $-2\log Q_{12}$ and $-2\log Q_{01}$.

We will need the following assumption, which is typical for this kind of theory. (The matrix $\beta_\perp$ is defined in a similar fashion as $\alpha_\perp$ above.) This assumption guarantees that X_t is an $I(1)$ process (c.f. Johansen, 1995, p.49).

Assumption A The roots of the characteristic equation corresponding to (2) have modulus > 1 or are equal to 1, and $\alpha'_{\perp}\Gamma\beta_{\perp}$ has full rank, where $\Gamma = I_{N_p} - \sum_{i=1}^{m-1}\Gamma_i$.

We now turn to the asymptotics of the test for cointegrating rank, which is used for the sequential rank estimation procedure. This is the test of $\mathcal{H}_2$ against $\mathcal{H}_4$. The main idea is to write $Q_{24} = Q_{23}Q_{34}$, implying

$$-2\log Q_{24} = -2\log Q_{23} - 2\log Q_{34}.$$

The result is that, as $T \to \infty$, $-2\log Q_{23}$ converges weakly to the χ^2 variate V, while $-2\log Q_{34}$ tends to U which has a Dickey-Fuller type distribution as given in the formulation of the theorem. Furthermore, $-2\log Q_{23}$ and $-2\log Q_{34}$ are asymptotically independent. Observe that if $r = 0$, the χ^2 variate disappears, and we have the usual Johansen trace test. Moreover, observe the short-hand notation of integrals, i.e. $\int FF' = \int_0^1 F(t)F(t)'\,dt$, etcetera.

Theorem 1 *Under $\mathcal{H}_2$, assumption A and if $\alpha'_{\perp}\mu \neq 0$ and $r > 0$, we have that as $T \to \infty$, the maximum likelihood ratio test of cointegrating rank r, Q_{24}, satisfies*

$$-2\log Q_{24} \xrightarrow{w} U + V,$$

where, defining $B(t)$ to be an $\{N(p-r)+1\}$-dimensional standard Wiener process (with mean zero and identity covariance matrix),

$$U = \operatorname{tr}\left\{\int dBF'\left(\int FF'\right)^{-1}\int FdB'\right\},$$

and where V is χ^2 with $N(N-1)(p-r)r$ degrees of freedom, independent of U. The process F is $\{N(p-r)+1\}$-dimensional with components

$$F_i(u) \equiv \begin{cases} B_i(u) - \int_0^1 B_i(t)\,dt, & i = 1, \dots, N(p-r), \\ u - \frac{1}{2}, & i = N(p-r)+1, \end{cases}$$

where the $B_i(t)$ are components of $B(t)$.

Proof. See the appendix. ■

The next theorem gives the asymptotic distribution of Q_{12}.

Theorem 2 *Under $\mathcal{H}_1$, assumption A and if $\alpha'_{\perp}\mu \neq 0$ and $r > 0$, we have that as $T \to \infty$, the maximum likelihood ratio test of common cointegrating space, Q_{12}, fulfills that $-2\log Q_{12}$ is asymptotically χ^2 with $(N-1)(p+1-r)r$ degrees of freedom.*

Proof. See the appendix. ■

In particular, in the PPP case, $-2\log Q_{12}$ is asymptotically χ^2 with $2(N-1)$ degrees of freedom.

Our final object is to test if, given cointegrating rank $r=1$ and $p=2$, the cointegrating relation is $\left(\beta_i', \beta_{N+1,i}'\right) = c_i(1,-1,1)$ for all i and constants c_i. This is a special case of $\mathcal{H}_0$ with $r=s=1$, all $H_i=(1,-1,1)'$ and all $\psi_i=c_i$.

Theorem 3 *Under $\mathcal{H}_0$, assumption A and if $\alpha_\perp'\mu \neq 0$ and $r>0$, we have that as $T\to\infty$, the maximum likelihood ratio test of the restriction $\left(\beta_i', \beta_{N+1,i}'\right)' = H_i\psi_i$ for all i, where the H_i are known $(p+1)\times s$ matrices and the ψ_i are $s\times r$ and unknown, against the hypothesis of common cointegrating space, Q_{01}, fulfills that $-2\log Q_{01}$ is asymptotically χ^2 with $(p+1-s)r$ degrees of freedom.*

Proof. See the appendix. ■

Note that in the PPP case, $-2\log Q_{01}$ is asymptotically $\chi^2(2)$.

4 The empirical analysis

Our database contains monthly observations of consumer prices and nominal exchange rates (versus the British pound) for Germany, France, and Italy for the years 1974 - 2000, i.e., $N=3$ and $T=314$. The Appendix contains a fuller description of the data and sources, as well as Figures 1-3, that show the consumer price series, the exchange rates, and the real exchange rates for these countries.

In the subsequent sections we use the framework outlined in previous sections in the following way: We begin by testing for the number of cointegrating relations in the 3 panels in a model that satisfies standard specification tests. We then carry on by testing hypotheses about the cointegration vectors; first by testing if the panel-specific cointegrating vectors span the same space, and if so, if the theoretical PPP-relationship holds for this space.

4.1 Specification and mis-specification analysis

The number of lags is specified using the information criterion proposed by Schwarz (1978), where an upper limit of five lags is pre-specified. The results suggest that $k=2$ is appropriate. Moreover, multivariate residual-based mis-specification tests with respect to serial correlation and non-Normality are found

$r =$	0	1	2
Test stat.	217	69.5	0.467
Crit. value	135	70.8	7.31

Table 1: Test statistics for cointegratin rank and bootstrapped critical values using 1,000 replicates for 5 % nominal test size.

	Germany	France	Italy
β_{ix}	1.00	1.00	1.00
β_{ip}	-5.12	9.87	-2.25
β_{ib}	0.671	-8.78	0.528

Table 2: Normalized unrestricted estimates of the cointegrating vectors.

to be insignificant at the 1%-level. The first-order Lagrange multiplier and the Box Ljung tests yield p-values of 0.181 and 0.027, and the Normality test a p-value of 0.05. Since these specification-tests are asymptotic and presumably affected by size-distortions in small-sample applications (c.f., Jacobson, Jansson, Vredin, and Warne (2001)), inference at the 1%-level rather than the usual 5%-level can be justified. Also, even if ARCH type errors are present in the data, asymptotic inference in cointegration models is robust, see Dennis, Hansen and Rahbek (2002). Given a lag-length of 2, the likelihood ratio tests for the three null hypotheses $r = 0, 1, 2$ are calculated against the alternative hypothesis of full rank. Complementary to the use of asymptotic distributions, we apply the method discussed above, i.e., a parametric bootstrap, as it was used in Gredenhoff and Jacobson (2001). By generating the bootstrap samples using the estimated empirical model (with the null hypothesis imposed) the resulting inference is robust with respect to parameter uncertainty. A nominal size of 5% is used and the number of bootstrap replicates is 1,000. The test statistics and corresponding critical values are given in Table (1) .We find that the null of $r = 0$ is rejected, while the null of $r = 1$ is not, hence, we conclude that there is one cointegrating relationship in each panel. The normalized cointegrating vectors are displayed in Table (2).

4.2 Testing linear restrictions

Having found support for the necessary condition for PPP, we now turn to the sufficient conditions. The multivariate setup used in this paper actually enables us to test for PPP in different ways. First, we test whether all three bilateral PPP relations span the same space, i.e., the three countries share the same economic laws but not necessarily the one outlined above. The test statistic is 17.4 with a bootstrapped critical value of 21.6 at a 5% nominal test level, and hence, we do not reject the null of a common cointegrating space. The normalized (with respect to β_{ix}) common cointegrating vector is $[1.00, -1.52, 0.885]'$ and has the correct signs and is thus remarkably close to the relationship implied by PPP. To test if PPP holds a likelihood ratio test with the cointegrating vector $[1, -1, 1]'$ as the null hypothesis is tested against common cointegrating space. The test statistic is found to be 60.8 with a bootstrapped critical value of 12.0, and thus we reject the null.

In summary, we have found support for our hypothesis that the variables in X_t can be characterized by an error correction model like equation (2). This implies that they are driven by a limited number of common stochastic trends and therefore are tied together in the long run. All three panels are characterized by one long-run, cointegrating, relation. However, although these long-run relations span the same cointegrating space and the estimated relationship is close to the theoretical one, a formal test rejects the hypothesis that they coincide.

5 Small-sample properties

Although we have used bootstrap based inference in the empirical section above, it is important to examine how the asymptotic distributions function in small samples. To analyze this a Monte Carlo simulation is performed. The data generating process (DGP) is the empirical model estimated in the previous section. We are interested in five different null hypotheses. The first three consider the rank: $r = 0$, 1 or 2, and the remaining two are tests on the cointegrating space: test of common space and test that the cointegrating vector is the theoretical PPP relationship, $[1, -1, 1]'$. The alternative for the first three models is the usual full rank model and for the last two an unrestricted cointegrating model with rank one. For the very last model the alternative of a common cointegrating space is also considered. The largest eigenvalues of the DGP's are displayed in Table (3). (See further the discussion below.)

The Monte Carlo setup is as follows. First we generate data according to the model under the null, then we estimate the models under the null and the alternative and calculate the likelihood ratio statistic. We then compare with the

$r = 0$	1	2
1.00	1.00	1.00
1.00	1.00	0.991
1.00	1.00	0.974
1.00	1.00	0.974
1.00	0.991	0.896
1.00	0.966	0.890
1.00	0.966	0.890
0.827	0.403	0.455
0.398	0.403	0.407
0.301	0.242	0.302
0.288	0.242	0.302
0.288	0.281	0.262
0.203	0.281	0.218
0.156	0.141	0.218

Table 3: Absolute values of the eigenvalues of the compagnion matrix for $r = 0, 1, 2$.

Null\T	100	200	314	400	800	1600
$r = 0$	0.625	0.271	0.159	0.134	0.078	0.082
$r = 1$	0.578	0.534	0.454	0.423	0.311	0.213
$r = 2$	0.494	0.487	0.256	0.142	0.074	0.086
Common$\|r = 1$	0.692	0.460	0.278	0.177	0.067	0.031
PPP$\|r = 1$ vs unrestricted	0.648	0.363	0.237	0.179	0.101	0.072
PPP$\|r = 1$ vs common	0.926	0.705	0.432	0.198	0.127	0.074

Table 4: Monte Carlo estimated test sizes for PPP related panel tests.

asymptotic critical value and note if the test rejects or does not reject the null. This is repeated 1,000 times and the proportion of rejections is the estimated size, which should be compared with a nominal size of 5%. The size adjusted power, i.e., the power when the simulated small sample critical values are used, is also of interest. For the null models with $r = 0$, 1 or 2, the DGP's have $r = 1$, 2 and full rank respectively. Regarding the cointegrating space tests, the DGP is the $r = 1$ model. We have also evaluated the power for a test of restrictions on the common cointegrating space. That is, the null hypothesis is given by the theoretical PPP-relationship, but the pseudo-data is generated from a model with the empirical estimates of the common cointegrating space imposed. The Monte Carlo simulations have been done for sample sizes $T = 100, 200, 400, 800$ and 1600, and the number of replicates is 1,000. The results are displayed in Table (4) and Table (5).

The results demonstrate the well-known problem in cointegration analysis

Null\T	100	200	314	400	800	1600
$r = 0$	0.688	0.999	1.000	1.000	1.000	1.000
$r = 1$	0.111	0.547	0.953	0.999	1.000	1.000
$r = 2$	0.018	0.022	0.168	0.404	0.921	0.988
Common$\|r = 1$	0.050	0.241	0.700	0.936	1.000	1.000
PPP$\|r = 1$ vs unrestricted	0.101	0.485	0.951	0.994	1.000	1.000
PPP$\|r = 1$ vs common	0.083	0.662	0.981	1.000	1.000	1.000

Table 5: Monte carlo estimated size adjusted power for PPP related panel tests.

that for larger systems, involving many parameters, the small sample critical values converge very slowly to the asymptotic ones (see e.g. Gredenhoff and Jacobson (2001)). In a panel setting similar to the one in the present paper, Larsson and Lyhagen (1999) obtain size problems for the test of cointegration rank, but not for tests of restrictions on the cointegrating vector. Here, however, in the rank-one model some eigenvalues of the companion matrix are very close to one in absolute value (see table (3)), indicating "closeness" to $I(2)$. This will make size problems even more severe, even for the restriction tests. This result clearly shows the need for some size-adjusting measure, such as a bootstrap test. The power properties are quite satisfactory for the larger sample sizes, $T = 400, 800, 1,600$. Hence, there is reason to assume that the empirical tests (based on $T = 314$) undertaken above do have reasonable power against the alternatives.

6 Conclusions

Using a multivariate cointegration panel data model this paper re-examines the evidence for PPP in post-Bretton Woods data for France, Germany, Italy and the United Kingdom. We find that each panel is characterized by exactly one cointegrating relation between nominal exchange rates and consumer prices, sometimes labelled as weak PPP, or the necessary condition for PPP to hold. Moreover, the panel-specific cointegrating vectors are found to span the same cointegrating space. We interpret this as support for PPP to the extent that the estimated long-run relations between exchange rates and consumer prices in our panels are, if not identical, at least closely related. But, testing for strong PPP, or the sufficient condition that the cointegrating space contains the vector of PPP-coefficients $(1, -1, 1)$, leads to rejection. It is, however, interesting to note that the over all panels estimated cointegrating vector is very close to the theoretical one; $(1, -1.5, 0.9)$, suggesting that PPP may, after all, be a reasonably accurate approximation of how nominal exchange rates and price levels evolve over time. This in turn suggests that it may not be important that economic models for long-term analysis allow for shocks to the real exchange rates with permanent effects.

These results have been estimated in a maximum likelihood cointegration panel data model with a specific structure that allows for the long-run dynamics implied by PPP. Specifically, the cointegrating matrix, β, is block diagonal with non-zero elements corresponding to each included country's nominal exchange rate and consumer prices. The numeraire country's consumer prices enter β through an extra row of non-zero elements, one for each panel. This particular structure requires new asymptotic results for the tests for cointegrating rank, a common cointegrating space over the panels, and restrictions on cointegrating vectors. Such limit distributions are derived in Section 3 and constitute the paper's main contribution. The results are extensions of the results of Johansen (1995) to the type of restrictions on β considered in this paper. In particular, we may note the asymptotic distribution of the test for cointegrating rank (Theorem 1), which is a sum of two independent variates, one being χ^2 and the other being Dickey-Fuller distributed. This result arises because we are testing reduced rank within the restricted β setting. As expected, the two other tests (Theorems 2 and 3) are asymptotic χ^2. These results follow as special cases of a theorem in Johansen (1991) on smooth hypotheses on β.

By means of Monte Carlo simulation we find that the asymptotic approximations are inadequate for smaller sample sizes, i.e., estimated test sizes are far from the nominal one and convergence occurs for samples of sizes we seldom have access to. The remedy for this size-distortion problem is a bootstrap test. By generating bootstrap samples using the estimated model as a DGP, we may estimate small-sample distributions. Now, whereas the bootstrap test can be expected to be approximately correct in size, it should be noted that its power will not be higher, nor lower, than the power of a size-adjusted asymptotic test. This has been theoretically predicted for the general case by Davidson and McKinnon (1996) and verified for the likelihood ratio test of linear restrictions on cointegrating vectors by Gredenhoff and Jacobson (1998) using Monte Carlo simulation. For this particular application we believe that the tests do have reasonable power judging by the results on size-adjusted power in the Monte Carlo experiments.

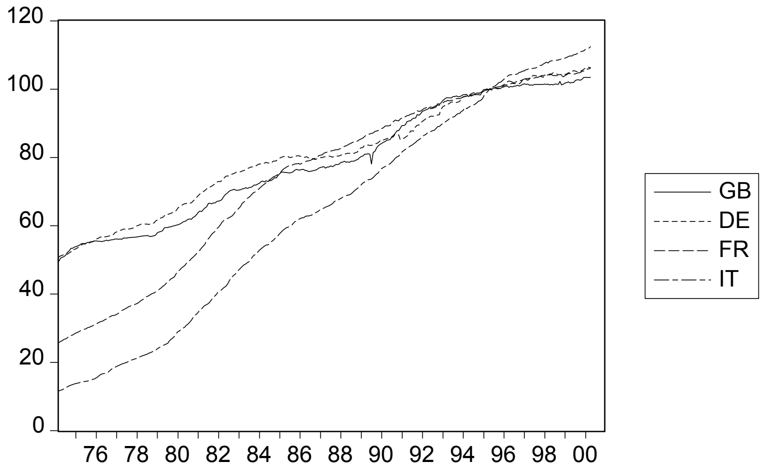

Figure 1: Monthly CPI for the United Kingdom, Germany, France and Italy

A Appendix

A.1 Description of data

The database is comprised of three nominal exchange rates and four consumer price indices. The frequency is monthly and the series run from 1974 to 1999, 1995 = 100. See Figures 1 -3. The exchange rates are given as the price of British pounds in German marks, French francs and Italian lire respectively. The consumer price indices are taken from row 63 in the IFS-tapes.

A.2 Omitted proofs

This section contains omitted proofs of Theorems 1-3. However, we start out by proving a theorem about the distribution of the estimated cointegrating space. This theorem and its proof is useful when proving Theorems 1-3. The proofs follow closely the proof of Theorem C.1 in Johansen (1991), which gives the corresponding result for any smooth hypothesis on β.

Figure 2: Monthly exchange rates agains the British pound for Germany, France and Italy

Figure 3: Monthly real exchange rate for Germany, France and Italy using the United Kingdom as the base country.

At first, define the $\{N(p+1)\} \times (Np+1)$ matrix

$$R \equiv \begin{pmatrix} (I_p,0) & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & (I_p,0) \\ (0,1) & \cdots & & (0,1) \end{pmatrix}'.$$

Then,

$$\beta = \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix} = R'\varphi,$$

where

$$\varphi \equiv \operatorname{diag}(\varphi_1, ..., \varphi_N),$$

with

$$\varphi_i \equiv \begin{pmatrix} \beta_i \\ \beta_{N+1,i} \end{pmatrix},$$

which are $(p+1) \times r$, for $i = 1, ..., N$. Thus, we may re-write (2) as

$$\Delta X_t = \alpha\varphi' R X_{t-1} + \sum_{i=1}^{m-1} \Gamma_i \Delta X_{t-i} + \mu + \varepsilon_t.$$

In the limit results needed in the sequel, this formulation enables us to use RX_{t-1} in place of X_{t-1}. Now, define $C \equiv \beta_\perp (\alpha'_\perp \Gamma \beta_\perp)^{-1} \alpha'_\perp$, where $\Gamma \equiv I_p - \sum_{i=1}^{m-1} \Gamma_i$. Granger's representation theorem (theorem 4.2 of Johansen (1995)) reads

Lemma 4 *If assumption A holds, we have the representation*

$$X_t = C\left(\mu t + \sum_{j=1}^{t} \varepsilon_j\right) + Y_t,$$

where Y_t is an $I(0)$ process.

Because of the lemma, the dominating deterministic trend of RX_t is $\tau \equiv RC\mu$, where $\Gamma \equiv I_{Np} - \sum_{i=1}^{m-1} \Gamma_i$. Hence, τ is $N(p+1) \times 1$, and we may write $\tau = (\tau'_1, ..., \tau'_N)'$, where for $i = 1, ..., N$, τ_i is $(p+1) \times 1$. Then, for each i, choose γ_i orthogonal to φ_i and τ_i. Then, γ_i is $p \times (p-r)$ and $\gamma \equiv \operatorname{diag}(\gamma_1, ..., \gamma_N)$

is orthogonal to φ and τ. Moreover, let $\{W(t)\}$ be a p-dimensional Wiener process with expectation 0 and covariance matrix Ω. The modified version of lemma A.2 of Johansen (1991) reads

Lemma 5 *As $T \to \infty$, we have for $u \in [0,1]$ that*

$$\begin{aligned} T^{-1/2}\overline{\gamma}' R X_{[Tu]} &\overset{w}{\to} \quad \overline{\gamma}' R C W(u), \\ T^{-1}\overline{\tau}' R X_{[Tu]} &\overset{w}{\to} \quad u. \end{aligned}$$

Now, put

$$\begin{aligned} \widetilde{G}_1(t) &\equiv \overline{\gamma}' R C \left\{ W(t) - \int_0^1 W(u)\, du \right\}, \\ \widetilde{G}_2(t) &\equiv t - \frac{1}{2}, \end{aligned}$$

$\widetilde{G}(t) \equiv \left\{ \widetilde{G}_1(t)', \widetilde{G}_2(t) \right\}'$, and $V(t) \equiv \alpha' \Omega^{-1} W(t)$. Observe that the $\widetilde{G}$ process is defined in a slightly different way than the G process in Johansen (1991), where there is no R matrix, and a bit different $\overline{\gamma}$ matrix. With this modification, the same limit results hold here, and we do not re-iterate them. Furthermore, define the matrix

$$S \equiv \left(H_1^{(r)} \otimes \widetilde{\widetilde{H}}_1, ..., H_N^{(r)} \otimes \widetilde{\widetilde{H}}_N \right)$$

with, for $i = 1, ..., N$ and n arbitrary,

$$H_i^{(n)} \equiv (0, ..., 0, I_n, 0, ..., 0)',$$

which is a $Nn \times n$ matrix with I_n as the ith block, and the $\widetilde{N} \times (p+1-r)$ matrix, where $\widetilde{N} \equiv N(p-r)+1$,

$$\widetilde{\widetilde{H}}_i \equiv \begin{pmatrix} H_i^{(p-r)} & 0 \\ 0 & 1 \end{pmatrix}.$$

This means that S is $Nr\widetilde{N} \times \kappa$, where $\kappa \equiv N(p+1-r)r$. We may now formulate our theorem. (From now on, we use short-hand notation for our integrals, i.e. $\int \widetilde{G}\widetilde{G}'$ instead of $\int_0^1 \widetilde{G}(t)\widetilde{G}(t)'\, dt$.)

Theorem 6 *As $T \to \infty$, the ML estimate of φ, $\widehat{\varphi}$, satisfies*

$$\left(T\gamma, T^{3/2}\tau \right)' \operatorname{vec}(\widehat{\varphi} - \varphi) \overset{w}{\to} S \left\{ S' \left(\alpha' \Omega^{-1} \alpha \otimes \int \widetilde{G}\widetilde{G}' \right) S \right\}^{-1} S' \operatorname{vec} \left\{ \int \widetilde{G}\, (dV)' \right\}.$$

Proof: As a preparation, write $\varphi = \operatorname{diag}(\varphi_1, ..., \varphi_N)$, where $\varphi_i = \left(\varphi_i^{(1)\prime}, \varphi_i^{(2)\prime}\right)^\prime$, where $\varphi_i^{(1)}$ is $r \times r$, for $i = 1, ..., N$. Then, it follows that $\alpha\varphi' = \alpha^*\varphi^{*\prime}$, where $\alpha^* \equiv \alpha \operatorname{diag}\left(\varphi_1^{(1)\prime}, ..., \varphi_N^{(1)\prime}\right)$ and $\varphi^* \equiv \operatorname{diag}(\varphi_1^*, ..., \varphi_N^*)$ with $\varphi_i^* \equiv (I_r, \vartheta_i')'$, $\vartheta_i \equiv \varphi_i^{(2)}\left\{\varphi_i^{(1)}\right\}^{-1}$ for $i = 1, ..., N$. (The ϑ_i are $(p+1-r) \times r$.) Hence, regarding $\varphi^* = \varphi^*(\vartheta)$ as a matrix-valued function of the elements of the vector $\vartheta \equiv \operatorname{vec}(\vartheta_1, \vartheta_2, ..., \vartheta_N)$, the derivatives are, denoting the elements of ϑ_i by ϑ_i^{jk}, where $j = 1, ..., p$, $k = 1, ..., r$, and similarly for φ,

$$\frac{\partial \varphi_{ii}^{*jk}}{\partial \vartheta_i^{lm}} = 1$$

if $\{(j,k) = (l,m)\}$, and 0 otherwise. (The derivatives of non diagonal block elements of φ^* w.r.t. any elements of ϑ are all zero.) Hence, the derivative in the direction $\widetilde{u}$, where $\widetilde{u} \equiv (u_1, ..., u_\kappa)'$ is a vector of the same structure as ϑ, i.e. $\kappa \times 1$ where $\kappa = N(p+1-r)r$, is the block diagonal $N(p+1-r) \times Nr$ matrix $D\varphi(\widetilde{u})$ with elements

$$\sum_{s=1}^{\kappa} u_s \frac{\partial \varphi_{ii}^{*jk}}{\partial \vartheta_s} = u_{s^*},$$

where s^* is the s corresponding to element (j,k) of the matrix ϑ_{ii}. In other words, $D\varphi(\widetilde{u})$ has the same structure as φ^*, except that the I_r matrices are replaced by 0.

Next, choose $\widetilde{u}_1, ..., \widetilde{u}_\kappa$ orthogonal in $\mathcal{R}^\kappa$, such that $\tau' D\varphi(\widetilde{u}_i) = 0$ for $i = 1, ..., \kappa_1$ and $\tau' D\varphi(\widetilde{u}_i) \neq 0$ for $i = \kappa_1 + 1, ..., \kappa$. Furthermore, define the matrix $D\widetilde{\varphi}$ with ith column given by

$$\begin{aligned} D\widetilde{\varphi}_i &= \operatorname{vec}\left\{(\gamma, 0)' D\varphi(\widetilde{u}_i)\right\}, \ i = 1, ..., \kappa_1, \\ D\widetilde{\varphi}_i &= \operatorname{vec}\left\{(0, \tau)' D\varphi(\widetilde{u}_i)\right\}, \ i = \kappa_1 + 1, ..., \kappa. \end{aligned}$$

Here, $(\gamma, 0)$ and $(0, \tau)$ are $Np \times \widetilde{N}$, and so, $(\gamma, 0)' D\varphi(\widetilde{u}_i)$ and $(0, \tau)' D\varphi(\widetilde{u}_i)$ are $\widetilde{N} \times Nr$ and $D\widetilde{\varphi}$ is $Nr\widetilde{N} \times \kappa$. To understand the structure of $D\widetilde{\varphi}$, write

$$\begin{aligned} \operatorname{vec}\left\{(\gamma, 0)' D\varphi(\widetilde{u}_i) I_{Nr}\right\} &= \left\{I_{Nr} \otimes (\gamma, 0)'\right\} \operatorname{vec}\left\{D\varphi(\widetilde{u}_i)\right\}, \\ \operatorname{vec}\left\{(0, \tau)' D\varphi(\widetilde{u}_i) I_{Nr}\right\} &= \left\{I_{Nr} \otimes (0, \tau)'\right\} \operatorname{vec}\left\{D\varphi(\widetilde{u}_i)\right\}, \end{aligned}$$

so that

$$D\widetilde{\varphi} = \left[\left\{I_{Nr} \otimes (\gamma, 0)'\right\} L_1, \left\{I_{Nr} \otimes (0, \tau)'\right\} L_2\right], \tag{4}$$

where

$$L_1 \equiv [\mathrm{vec}\{D\varphi(\widetilde{u}_1)\},...,\mathrm{vec}\{D\varphi(\widetilde{u}_{\kappa_1})\}],$$
$$L_2 \equiv [\mathrm{vec}\{D\varphi(\widetilde{u}_{\kappa_1+1})\},...,\mathrm{vec}\{D\varphi(\widetilde{u}_{\kappa})\}].$$

Next, for $i = 1,...,N$ define $H_i^{(r)}$ as above, which is orthogonal to the $Nr \times (N-1)r$ matrix

$$H_{i\perp}^{(r)} = \begin{pmatrix} I_{r(i-1)} & 0_{r(i-1)\times r(N-1-i)} \\ 0_{r\times r(i-1)} & 0_{r\times r(N-1-i)} \\ 0_{r(N-1-i)\times r(i-1)} & I_{r(N-1-i)} \end{pmatrix},$$

for $i = 1,...,N$. Furthermore, let

$$\widetilde{H}_i \equiv \begin{pmatrix} H_i^{(p-r)} \\ 0 \end{pmatrix},$$

which is $\widetilde{N} \times (p-r)$, and

$$S_\perp \equiv \left(H_{1\perp}^{(r)} \otimes \widetilde{H}_1,...,H_{N\perp}^{(r)} \otimes \widetilde{H}_N\right),$$

which is $Nr\widetilde{N} \times N(N-1)(p-r)r$ and orthogonal to S defined above. Now, we find via (4) that

$$\begin{aligned}
& S_\perp' \left\{I_{Nr} \otimes (\gamma,0)'\right\} L_1 \\
= & \begin{pmatrix} H_{1\perp}^{(r)\prime} \otimes \widetilde{H}_1'(\gamma,0)' \\ \vdots \\ H_{N\perp}^{(r)\prime} \otimes \widetilde{H}_N'(\gamma,0)' \end{pmatrix} [\mathrm{vec}\{D\varphi(\widetilde{u}_1)\},...,\mathrm{vec}\{D\varphi(\widetilde{u}_{\kappa_1})\}], \\
= & \begin{pmatrix} \mathrm{vec}\left(\widetilde{H}_1'(\gamma,0)' D\varphi(\widetilde{u}_1) H_{1\perp}^{(r)}\right) & \cdots & \mathrm{vec}\left(\widetilde{H}_1'(\gamma,0)' D\varphi(\widetilde{u}_{\kappa_1}) H_{1\perp}^{(r)}\right) \\ \vdots & & \vdots \\ \mathrm{vec}\left(\widetilde{H}_N'(\gamma,0)' D\varphi(\widetilde{u}_1) H_{N\perp}^{(r)}\right) & \cdots & \mathrm{vec}\left(\widetilde{H}_N'(\gamma,0)' D\varphi(\widetilde{u}_{\kappa_1}) H_{N\perp}^{(r)}\right) \end{pmatrix} = 0,
\end{aligned}$$

where the third equality follows since for all i and j, $\widetilde{H}_i'(\gamma,0)' D\varphi(\widetilde{u}_j) H_{i\perp}^{(r)}$ picks out only non-diagonal blocks of $(\gamma,0)' D\varphi(\widetilde{u}_j)$, which are 0. Similarly, $S_\perp' \left\{I_{Nr} \otimes (0,\tau)'\right\} L_2 = 0$. Now, let S be as above. Then, the identity

$$I_{Nr\widetilde{N}} = S\overline{S}' + \overline{S}_\perp S_\perp'$$

and (4) imply

$$D\widetilde{\varphi} = \left(S\overline{S}' + \overline{S}_\perp S_\perp'\right)\left[\left\{I_{Nr} \otimes (\gamma,0)'\right\} L_1, \left\{I_{Nr} \otimes (0,\tau)'\right\} L_2\right] = SM,$$

where

$$M \equiv \overline{S}' \left[\left\{ I_{Nr} \otimes (\gamma, 0)' \right\} L_1, \left\{ I_{Nr} \otimes (0, \tau)' \right\} L_2 \right],$$

which is a $\kappa \times \kappa$ matrix. Now, if M is non-singular, eq. (C.7) of Johansen (1991) yields (with the Kronecker product twisted around, due to different notational conventions),

$$\begin{aligned}
& \left(T\gamma, T^{3/2}\tau\right)' \operatorname{vec}(\widehat{\varphi} - \varphi) \\
& \xrightarrow{w} D\widetilde{\varphi} \left\{ D\widetilde{\varphi}' \left(\alpha'\Omega^{-1}\alpha \otimes \int \widetilde{G}\widetilde{G}' \right) D\widetilde{\varphi} \right\}^{-1} D\widetilde{\varphi}' \operatorname{vec} \left\{ \int \widetilde{G}\,(dV)' \right\} \\
= \;\; & SM \left\{ M'S' \left(\alpha'\Omega^{-1}\alpha \otimes \int \widetilde{G}\widetilde{G}' \right) SM \right\}^{-1} M'S' \operatorname{vec} \left\{ \int \widetilde{G}\,(dV)' \right\} \\
= \;\; & S \left\{ S' \left(\alpha'\Omega^{-1}\alpha \otimes \int \widetilde{G}\widetilde{G}' \right) S \right\}^{-1} S' \operatorname{vec} \left\{ \int \widetilde{G}\,(dV)' \right\},
\end{aligned}$$

as was to be shown. The non-singularity of M follows since

$$\begin{aligned}
& S' \left\{ I_{Nr} \otimes (\gamma, 0)' \right\} L_1 \\
= \;\; & \begin{pmatrix} H_1^{(r)\prime} \otimes \widetilde{H}_1' (\gamma, 0)' \\ \vdots \\ H_N^{(r)\prime} \otimes \widetilde{H}_N' (\gamma, 0)' \end{pmatrix} \left[\operatorname{vec}\left\{ D\varphi(\widetilde{u}_1) \right\}, ..., \operatorname{vec}\left\{ D\varphi(\widetilde{u}_{\kappa_1}) \right\} \right], \\
= \;\; & \begin{pmatrix} \operatorname{vec}\left(\widetilde{H}_1' (\gamma, 0)' D\varphi(\widetilde{u}_1) H_1^{(r)} \right) & \cdots & \operatorname{vec}\left(\widetilde{H}_1' (\gamma, 0)' D\varphi(\widetilde{u}_{\kappa_1}) H_1^{(r)} \right) \\ \vdots & & \vdots \\ \operatorname{vec}\left(\widetilde{H}_N' (\gamma, 0)' D\varphi(\widetilde{u}_1) H_N^{(r)} \right) & \cdots & \operatorname{vec}\left(\widetilde{H}_N' (\gamma, 0)' D\varphi(\widetilde{u}_{\kappa_1}) H_N^{(r)} \right) \end{pmatrix}
\end{aligned}$$

which is a block diagonal matrix with Nr diagonal blocks of dimension $(p+1-r) \times (p+1-r)$. For example, the first block is given by

$$\begin{aligned}
& \left(\operatorname{vec}\left(\widetilde{H}_1' (\gamma, 0)' D\varphi(\widetilde{u}_1) H_1^{(r)} \right), ..., \operatorname{vec}\left(\widetilde{H}_1' (\gamma, 0)' D\varphi(\widetilde{u}_{p+1-r}) H_1^{(r)} \right) \right) \\
= \;\; & (\gamma_1', 0)\,(0, I_{p+1-r})'.
\end{aligned}$$

Indeed, this is the form of the r uppermost blocks, then comes the block $(\gamma_2', 0)\,(0, I_{p+1-r})'$, etcetera. Similarly, $S' \left\{ I_{Nr} \otimes (0, \tau)' \right\} L_2$ is block diagonal with Nr diagonal blocks of dimension $(p+1-r) \times (p+1-r)$, given by $(0, \tau_1)\,(0, I_{p+1-r})'$, etcetera. Hence,

$$M\,(I_{\kappa_1}, I_{\kappa-\kappa_1})' = \left[S' \left\{ I_{Nr} \otimes (\gamma, 0)' \right\} L_1, S' \left\{ I_{Nr} \otimes (0, \tau)' \right\} L_2 \right] (I_{\kappa_1}, I_{\kappa-\kappa_1})'$$

is block diagonal with Nr diagonal blocks of dimension $(p+1-r) \times (p+1-r)$, given by $(\gamma_1, \tau_1)\,(0, I_{p+1-r})'$, etcetera. Since these blocks are non-singular, it follows that $M\,(I_{\kappa_1}, I_{\kappa-\kappa_1})'$, and hence M, is non-singular.

A.2.1 Proof of theorem 1

Consider the three hypotheses $\mathcal{H}_4$: rank$(\Pi) \leqslant Np$, $\mathcal{H}_3 : \Pi = \alpha\beta'$ where α, β are $Np \times Nr$, of full rank and $\mathcal{H}_2$: as $\mathcal{H}_3$ but where β is block-diagonal with $p \times r$-dimensional blocks. Denoting the maximum likelihood ratio between $\mathcal{H}_i$ and $\mathcal{H}_j$ ($\mathcal{H}_i \subset \mathcal{H}_j$) by Q_{ij}, we then have $Q_{24} = Q_{23}Q_{34}$, i.e.

$$-2\log Q_{24} = -2\log Q_{23} - 2\log Q_{34}.$$

Johansen (1995) has showed that the asymptotic distribution of $-2\log Q_{34}$ equals the distribution of U as defined in the theorem. (The fact that β has the specific form under our hypothesis under test, $\mathcal{H}_2$, does not affect this result.) Now, to prove our theorem, our plan is

1) To show the convergence of $-2\log Q_{23}$ to the χ^2 distribution.

2) To show the asymptotic independence between $-2\log Q_{23}$ and $-2\log Q_{34}$.

1) It follows from theorem C.1 of Johansen (1991) that $-2\log Q_{23}$ is asymptotically $\chi^2 (k - s)$, where k is the number of free parameters under the alternative hypothesis $\mathcal{H}_3$, and where s is the number of free parameters under the null hypothesis $\mathcal{H}_2$. In other words, $k - s$ is the difference between the numbers of free parameters of $\alpha\beta'$ under $\mathcal{H}_3$ and $\mathcal{H}_2$, respectively. Now, under $\mathcal{H}_2$,

$$\begin{aligned}
\beta &= \begin{pmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \beta_N \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \beta_1^{(1)} \\ \beta_1^{(2)} \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} \beta_N^{(1)} \\ \beta_N^{(2)} \end{pmatrix} \\ \beta_{N+1,1} & \cdots & & \beta_{N+1,N} \end{pmatrix} \\
&= \widetilde{\beta}\,\mathrm{diag}\left(\beta_1^{(1)}, ..., \beta_N^{(1)}\right),
\end{aligned}$$

where

$$\widetilde{\beta} \equiv \begin{pmatrix} \widetilde{\beta}_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \widetilde{\beta}_N \\ \widetilde{\beta}_{N+1,1} & \cdots & & \widetilde{\beta}_{N+1,N} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} I_r \\ \vartheta_1 \end{pmatrix} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & & 0 \\ 0 & \cdots & 0 & \begin{pmatrix} I_r \\ \vartheta_N \end{pmatrix} \\ \vartheta_{N+1,1} & \cdots & & \vartheta_{N+1,N} \end{pmatrix}, \tag{5}$$

with $\vartheta_i \equiv \beta_i^{(2)} \left\{\beta_i^{(1)}\right\}^{-1}$ and $\vartheta_{N+1,i} \equiv \beta_{N+1,i}^{(2)} \left\{\beta_{N+1,i}^{(1)}\right\}^{-1}$ for $i = 1, ..., N$. (The $\beta_i^{(1)}$ are $r \times r$ and the $\beta_i^{(2)}$ are $(p-r) \times r$.) Then, $\alpha\beta' = \widetilde{\alpha}\widetilde{\beta}'$, where $\widetilde{\alpha} \equiv \alpha \operatorname{diag}\left(\beta_1^{(1)\prime}, ..., \beta_N^{(1)\prime}\right)$. Here, for each i, the numbers of free parameters of $\widetilde{\beta}_i$ and $\widetilde{\beta}_{N+1,i}$ are $(p-r)\,r$ and r, respectively. Consequently, under $\mathcal{H}_2$, the number of free parameters of $\alpha\beta'$ is $N^2pr + N\,(p-r)\,r + Nr$. Under $\mathcal{H}_3$, the corresponding trick yields $N^2pr + N^2\,(p-r)\,r + Nr$ free parameters. Hence,

$$\begin{aligned} k - s &= \left\{N^2pr + N^2\,(p-r)\,r + Nr\right\} - \left\{N^2pr + N\,(p-r)\,r + Nr\right\} \\ &= N\,(N-1)\,(p-r)\,r, \end{aligned}$$

as was to be shown.

2) Introduce the extra hypothesis $\mathcal{H}_0 : \Pi = \alpha\beta'$ where β is fixed. It follows as above that

$$-2\log Q_{23} = -2\log Q_{03} - (-2\log Q_{02}).$$

Here, following Johansen (1991), p. 1576, we find as in the preceding proof that

$$\begin{aligned} & -2\log Q_{02} \\ & \xrightarrow{w} \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}' D\widetilde{\beta}\left\{D\widetilde{\beta}'\left(\alpha'\Omega^{-1}\alpha \otimes \int \widetilde{G}\widetilde{G}'\right) D\widetilde{\beta}\right\}^{-1} D\widetilde{\beta}' \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\} \\ = \; & \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}' SM\left\{M'S'\left(\alpha'\Omega^{-1}\alpha \otimes \int \widetilde{G}\widetilde{G}'\right) SM\right\}^{-1} M'S' \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\} \\ = \; & \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}' S\left\{S'\left(\alpha'\Omega^{-1}\alpha \otimes \int \widetilde{G}\widetilde{G}'\right) S\right\}^{-1} S' \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}. \end{aligned}$$

As for $-2\log Q_{03}$, we simply replace $D\widetilde{\beta}$ by an identity matrix, to obtain

$$\begin{aligned}
&-2\log Q_{03}\\
&\xrightarrow{w} \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}'\left(\alpha'\Omega^{-1}\alpha\otimes\int\widetilde{G}\widetilde{G}'\right)^{-1}\operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}\\
= \quad &\operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}'\left(\alpha'\Omega^{-1}\alpha\otimes\int\widetilde{G}\widetilde{G}'\right)^{-1}\operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}.
\end{aligned}$$

Consequently,

$$-2\log Q_{23} \xrightarrow{w} \operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}' P\operatorname{vec}\left\{\int \widetilde{G}\,(dV)'\right\}, \tag{6}$$

where

$$P\equiv J^{-1}-S\,(S'JS)^{-1}\,S'$$

with

$$J\equiv\alpha'\Omega^{-1}\alpha\otimes\int\widetilde{G}\widetilde{G}'.$$

Then, it follows that

$$P=J^{-1}S_{\perp}\left(S_{\perp}'J^{-1}S_{\perp}\right)^{-1}S_{\perp}'J^{-1}. \tag{7}$$

Now, conditional on $\widetilde{G}$, $\operatorname{vec}\left(\int\widetilde{G}dV'\right)$ is normal with expectation 0 and covariance matrix J, and so, $S_{\perp}'J^{-1}\operatorname{vec}\left(\int\widetilde{G}dV'\right)$ is normal with expectation 0 and covariance matrix $S_{\perp}'J^{-1}S_{\perp}$. Hence, by (7), the r.h.s. of (6) is $\chi^2(1)$, conditional on $\widetilde{G}$. Moreover, since this distribution is independent of $\widetilde{G}$, this property holds also unconditionally. Consequently, the quantity on right-hand side of (6) is independent of $\widetilde{G}$, a fact that will be useful below. Furthermore, from Johansen (1995), p. 158-160, we deduce the representation

$$\begin{aligned}
&-2\log Q_{34}\\
&\xrightarrow{w} \operatorname{tr}\left\{\left(\int GG'\right)^{-1}\int GdW'\alpha_{\perp}\left(\alpha_{\perp}'\Omega\alpha_{\perp}\right)^{-1}\alpha_{\perp}'\left(\int GdW'\right)'\right\}\\
= \quad &\operatorname{vec}\left(\int GdW'\alpha_{\perp}\right)'\left(\alpha_{\perp}'\Omega\alpha_{\perp}\otimes\int GG'\right)^{-1}\operatorname{vec}\left(\int GdW'\alpha_{\perp}\right) \qquad (8)
\end{aligned}$$

where W is as above. The processes G and $\widetilde{G}$ are defined in slightly different ways, but the stochastic parts of them are the same. In fact, defining

$$Z(t)\equiv\left(W(t)'-\int_0^1 W(u)'\,du,\quad t-\frac{1}{2}\right)',$$

we have $\widetilde{G}(t) = \widetilde{A}Z(t)$, where $\widetilde{A} \equiv \text{diag}(\overline{\gamma}' RC,\ 1)$. Similarly, $G(t) = AZ(t)$, where A is as $\widetilde{A}$, but with no R and a slightly different $\overline{\gamma}$. Now, we need to show that the right hand side terms of (6) and (8), M_1 and M_2 say, are independent. To this end, let us condition on Z. Then, $\int \widetilde{G}(dV)' = \widetilde{A} \int ZdW'\Omega^{-1}\alpha$ and $\int GdW'\alpha_\perp = A \int ZdW'\alpha_\perp$ are both normals, each with expectation zero, and the covariance between them is

$$\widetilde{A}\mathrm{E}\left\{\int ZdW'\Omega^{-1}\alpha\left(\int ZdW'\alpha_\perp\right)'\right\}A' = 0,$$

showing that $\int \widetilde{G}(dV)'$ and $\int GdW'\alpha_\perp$ are conditionally independent given Z. Hence, M_1 and M_2 must also be conditionally independent given Z. Furthermore, as we saw above, M_1 is independent of G, hence also of Z. Thus we get, denoting the densities for M_1 and M_2 by f_1 and f_2, their simultaneous density by $f_{1,2}$, the density of Z by f_Z and the corresponding conditional densities by $f_{1|Z}$ etcetera,

$$f_{1,2} = \int f_{1,2|Z}f_Z = \int f_{1|Z}f_{2|Z}f_Z = f_1 \int f_{2|Z}f_Z = f_1 f_2,$$

where the integrals are over the support of the Z density. This shows the independence between M_1 and M_2, and we are done.

A.2.2 Proof of theorem 2

Again, the theorem is a special case of theorem C.1 of Johansen (1991). Hence, the asymptotic distribution is χ^2, and the number of degrees of freedom is the difference between the number of free parameters of $\alpha\beta'$ under $\mathcal{H}_2$ and $\mathcal{H}_1$, respectively. As we saw in the previous proof, the former number is $N^2pr + N(p+1-r)r$. Similar arguments lead to the corresponding number $N^2pr + (p+1-r)r$ under $\mathcal{H}_1$. Consequently, the number of degrees of freedom for the test is the difference, $(N-1)(p+1-r)r$, as was to be shown.

A.2.3 Proof of theorem 3

As above, the asymptotic distribution is χ^2. We need to find the number of degrees of freedom. Now, under $\mathcal{H}_0$, we may without loss of generality assume that all ψ_i are equal. Consequently, with $\psi \equiv (\psi_0', \psi_1')'$ where ψ_0 is $r \times r$, we may write

$$
\begin{aligned}
\beta &= \left(\begin{pmatrix} I_p & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_{N+1,1} \end{pmatrix}, ..., \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ I_p & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_N \\ \beta_{N+1,N} \end{pmatrix} \right) \\
&= \left(\begin{pmatrix} I_p & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix} H_1, ..., \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ I_p & 0 \\ 0 & 1 \end{pmatrix} H_N \right) \left\{ I_N \otimes \begin{pmatrix} I_r \\ \psi_1 \psi_0^{-1} \end{pmatrix} \right\} (I_N \otimes \psi_0) \\
&\equiv \beta^* (I_N \otimes \psi_0),
\end{aligned}
$$

implying $\alpha\beta' = \alpha^*\beta^{*\prime}$, where $\alpha^* \equiv \alpha \left(I_N \otimes \psi_0'\right)$. Here, β^* has $(s-r)\,r$ free parameters, and so, the number of free parameters of $\alpha\beta'$ is $N^2pr + (s-r)\,r$. Hence, in the manner as in the previous proofs, the number of degrees of freedom for the test of $\mathcal{H}_0$ against $\mathcal{H}_1$ is

$$
N^2pr + (p+1-r)\,r + r - \left\{N^2pr + (s-r)\,r\right\} = (p+1-s)\,r.
$$

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