Average Inflation Targeting*

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Abstract

The analysis of this paper demonstrates that when the Phillips curve has forward-looking components, a goal for *average* inflation — i.e. targeting a j-period average of one-period inflation rates — will cause inflation expectations to change in a way that improves the short-run trade-off faced by the monetary policymaker. Average inflation targeting is thus an example of a 'modified' loss function, which when implemented in a discretionary fashion results in more efficient outcomes from the standpoint of the true social objective (inflation targeting under commitment), than the discretionary pursuit of the true objective itself. In purely forward-looking models, average inflation targeting is dominated by price level targeting. But we also demonstrate that in models where the Phillips curve has both forward- and backward-looking components, there are cases when the average inflation target provides more efficient outcomes than both 'ordinary' one-period inflation targeting and price level targeting.

Key words: optimal monetary policy, inflation targeting, optimal delegation.

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1 Introduction

The past decade has seen a substantial reduction in inflation rates in the industrialized world. While the average annual rate of inflation among the eleven major industrialized countries was over 7.5 percent in 1973-1987, it fell to below 3 percent in 1988-1999.[1] With this reduction in inflation rates the meaning of 'price stability' has changed. During the high-inflation period, the strive for price stability most often meant simply an attempt to curtail the rate of *increase* in the price level. A stable price level was considered unattainable (and thus its desireability was not debated either). Now, in a low-inflation environment, calls for price stability increasingly mean exactly that, i.e. a stable *level* of prices.

Recent years has thus seen a renewed interest in the analysis of monetary policy geared towards keeping a stable level of prices. One strand of the literature has shed new light on the question of the relative merits of price level targeting and inflation targeting (see e.g. Dittmar and Gavin 2000, Kiley 1998, Svensson 1999 and Vestin 2000.) It has been demonstrated that a price level target can give a more favorable combination of variability in inflation and output gap than an inflation targeting policy, when the central bank is constrained to act under discretion. In particular, Vestin (2000) shows in a forward-looking model that price level targeting under discretion comes closer to mimicking a socially optimal policy of inflation targeting under commitment, than does inflation targeting under discretion. The basic reason behind this result is that a price level target will make inflation expectations change in such a way to help the monetary policy maker; policy does not have to react as strongly as otherwise. More formally, price level targeting introduces *history dependence*, which is a characteristic of commitment solutions as shown by Woodford (1999b).

In this paper we investigate a set of policies that, as we argue below, may be considered as lying between price level targeting and inflation targeting, namely *average* inflation targeting. By this we mean a policy where the central bank's objective is to keep average inflation measured over several years stable. There are several reasons why we believe this

[1] Unweighted average for the G10 countries plus Switzerland. Source: IFS.

may be interesting. First, average inflation targeting also introduces history dependence, but to a varying degree, depending on the width of the window used when calculating the average inflation rate.[2] Therefore it is of theoretical interest to see if average inflation targeting under discretion produces policy responses closer to the optimal, commitment responses. Second, since there is (at least) one central bank pursuing an average inflation target, the Reserve Bank of Australia, such a policy deserves to be analyzed.[3,4]

The results of the paper are the following. Most importantly, we show that having an average inflation target results in inflation expectations adapting in such a way that will improve the short-run trade-off faced by the monetary policymaker, leading to smoother policy responses. Thus, given the assumption that the optimal commitment solution cannot be achieved (e.g. due to institutional constraints), average inflation targeting is an example of a 'modified' loss function, which when pursued in a discretionary setting produces more efficient outcomes than the discretionary pursuit of the true objective.[5] In a purely forward-looking model, such average inflation targets are typically dominated by a price level target (i.e. the price level targeting case comes closer to the optimal, commitment solution). But we also show that in a model with both forward- and backward-looking components, there are cases where an average inflation target provides more efficient outcomes than a price level target.

The remainder of this paper is structured as follows. In Section 2 we discuss societal preferences and different alternatives for the loss function assigned to the central bank. In the following section we compare the properties of a price-level targeting regime and inflation targeting regimes with varying definitions of 'inflation' in a very simple, wholly forward-looking model of the economy. Section 4 examines a 'hybrid' model, one which accomodates both backward-looking and forward-looking behavior, and which provides

[2] See e.g. Nessén (1999).

[3] One could perhaps also argue that the ECB, with its emphasis on monetary stability in the medium term, is another example of a central bank (implicitly) pursuing an average inflation target.

[4] See also King (1999) for a discussion on average inflation targeting.

[5] See Woodford (1999b) and Svensson and Woodford (1999) for a discussion. Other examples of such modified loss functions include nominal income growth targeting (Jensen, 1999) inflation-and-monetary targeting (Söderström, 2000) and inflation targeting *cum* interest rate smoothing (Woodford, 1999a).

some indication of the robustness of the results from Section 3.[6] The final section concludes. Most of the mathematical derivations have been put in the Appendices.

2 Societal preferences, commitment, and delegation to the central bank

Following Woodford (1999b), let society's preferences be modelled as

$$(1-\beta)\sum_{\tau=t}^{\infty} \beta^{\tau-t} L^*\left(\pi_\tau, x_\tau\right), \tag{1}$$

where π_t is inflation between periods $t-1$ and t (i.e $p_t - p_{t-1}$), x_t is the output gap at time t, and β is a discount factor ($0 < \beta \leq 1$). The society period loss function is defined as

$$L^*\left(\pi_t, x_t\right) \equiv \frac{1}{2}\left[\left(\pi_t - \pi^*\right)^2 + \lambda\, x_t^2\right], \tag{2}$$

λ being the relative weight on output stabilization versus inflation stabilization. Loss functions of this form are very common in analyses of monetary policy, since they are believed to capture the salient features of inflation targeting regimes: inflation is stabilized around a target π^*, while output is stabilized around the natural level; and the deviations of inflation and output from their respective targets are squared, making deviations equally undesireable regardless of sign (i.e. the inflation and output targets are symmetric). Such loss functions have habitually been introduced into models of optimal monetary policy in an *ad hoc* manner. But, as recently shown by Woodford (1999a), (2) can be obtained as a second-order Taylor series approximation to the expected utility level of the representative household in a theoretical model of the sort used in this paper, i.e. it is an approximation of the theoretically correct welfare measure.

A critical assumption in the analysis of optimal monetary policy is whether or not the central bank can act under commitment. A famous result due to Barro and Gordon (1983) is that an *inflation bias* may arise if the central bank has a goal for output in excess of the natural level, and is unable to commit to future paths of policy. More recent results regarding monetary policy in forward-looking models (which are the kinds

[6] For robustness-checks with respect to credibility issues and uncertainty, see Yetman (2000).

of models that will be analyzed in this paper) stress that a so-called *stabilization bias* may arise, independent of the central bank's output goal (see Backus and Driffil, 1986, Currie and Levine, 1993, Clarida, Gali and Gertler, 1999 and Woodford 1999a, 1999b, 2000). What this means is that the short-run response of monetary policy to shocks will differ depending upon whether the central bank acts under commitment or discretion. Under commitment, a central bank can benefit today from the anticipation of future policy actions. Thus a solution under commitment gives the socially optimal outcome in the sense of producing the lowest discounted sum of losses; the discretionary solution will result in a higher overall loss (see Woodford, 1999b, or Appendix C in this paper for a further discussion).

Yet, there are strong reasons to believe that commitment is not a realistic assumption to make regarding the way in which policy is conducted. Thus the point of departure in this paper is that only discretionary policies belong to the set of feasible policies. The question we subsequently study can be posed in the following manner. In the absence of a commitment technology, can the discretionary solution be improved upon by assigning another, 'modified' loss function to an independent central bank?

More specifically, we envision the central bank as being assigned the task of minimizing the infinite discounted sum of period loss functions

$$\min_{\{i_\tau\}_{\tau=t}^{\infty}} \; \mathrm{E}_t \, (1-\beta) \sum_{\tau=t}^{\infty} \beta^{\tau-t} L_\tau, \tag{3}$$

where β is the same discount factor as before and i_t is the central bank instrument (to be defined) at time τ. The question is, how should society design the mandate that it delegates to the central bank, i.e. what should be period loss function L_t be?[7] One candidate is perhaps (2), i.e. the central bank is instructed to target the annual inflation rate. But as already mentioned, it is the suboptimal properties of the solution generated under this loss function that motivates the search for another, 'modified' loss function. Another possible candidate, one which has received much attention in the the recent

[7]The maintained assumption throughout is that the delegation of this loss function is perfectly credible. It is only with regard to the *implementation* of a given loss function that credibility issues arise.

literature, is a price level target:

$$L^{PT}(p_t, x_t) \equiv \frac{1}{2}\left[(p_t - p^*)^2 + \widetilde{\lambda}\, x_t^2\right], \tag{4}$$

where $\widetilde{\lambda}$ is the relative weight on output stabilization versus price level stabilization (see the references mentioned in the Introduction). In this paper we examine these two alternatives, inflation targeting and price level targeting, plus a spectrum of 'intermediate' regimes. These intermediate regimes correspond to the central bank being be instructed to target the j-period average inflation rate:

$$L^{ITj}(\overline{\pi}_{j,t}, x_t) \equiv \frac{1}{2}\left[(\overline{\pi}_{j,t} - \pi^*)^2 + \overline{\lambda}\, x_t^2\right], \tag{5}$$

where the preference parameter $\overline{\lambda}$ is the relative weight on output stabilization versus j-period average inflation stabilization, the *average* inflation rate $\overline{\pi}_{j,t}$ being defined as

$$\begin{aligned} \overline{\pi}_{j,t} &\equiv \frac{1}{j}\sum_{s=0}^{j-1} \pi_{t-s} \\ &= \frac{1}{j}(p_t - p_{t-j}). \end{aligned} \tag{6}$$

In what way may average inflation targeting be seen as a sequence of intermediate regimes, lying inbetween inflation targeting and price level targeting? If we let $j = 1$ in (6) then we of course have the 'conventional', one-period inflation target. Further, the price level at any point in time may be expressed as the sum of the history of inflation rates plus an initial price level, or

$$p_t - p_0 = \pi_t + \pi_{t-1} + \pi_{t-2} + \pi_{t-3} + \ldots + \pi_1.$$

Thus, letting j become very large in (6) corresponds roughly (up to a proportionality factor) to having a price level target. Later on in this paper, we will explore how large j must be in order for the resulting policies to be roughly the same as the policies obtained with a price level target.[8]

[8] Batini and Yates (2000) have an alternative way of formulating the intermediate regimes. They specify a loss function where a convex combination of inflation and the price level is one of the arguments, along with the output gap.

3 A simple forward-looking model

We begin by examining the relative properties of policies implied by the different loss functions (2), (4), and (5) in a very simple economy, consisting of what is commonly referred to as a "New-Keynesian" Phillips curve linking the inflation rate π_t to the output gap x_t and expected future inflation $\pi_{t+1|t}$:[9]

$$\pi_t = \beta \pi_{t+1|t} + \kappa x_t + u_t, \tag{7}$$

where u_t is an exogenous shock, κ is a positive coefficient and β is the same discount factor as before. This equation can be derived as the log-linear approximation to the first-order conditions for optimal price-setting in an economic environment with monopolistically competitive firms and sticky prices[10], and has been analyzed extensively by Clarida, Gali and Gertler (1999) (henceforth CGG) and Woodford (1999b). Since prices cannot be continuously changed, firms take into consideration expectations of future marginal costs, in addition to current conditions, when setting their optimal price.[11] Variations in marginal costs due to variations in excess demand are captured by the term involving the output gap in (7). The shock u_t may be labelled a cost-push shock, since it captures everything else (other than demand conditions) that affects expected future marginal costs. We introduce so-called exogenous persistence by letting this exogenous cost-push shock follow an AR(1) process:

$$u_t = \rho u_{t-1} + \varepsilon_t, \tag{8}$$

where $0 \leq \rho < 1$.

For analytical simplicity, we assume that the output gap, x_t, is the instrument of the central bank. Typically, in analyses of monetary policy with inflation targets, an aggregate demand relation is also specified, linking the interest rate (i.e. the instrument in such models) to the output gap. However, as long as there are no separate concerns about interest rate smoothing in the objective function such an equation is redundant for

[9] $\pi_{t+1|t}$ is shorthand notation for $\mathrm{E}_t(\pi_{t+1})$.

[10] More specifically, the sticky prices are modelled according to Calvo (1983), whereby the opportunity for a firm to change its price arrives stochastically and exogenously.

[11] This can be seen most clearly by iterating forward on (7), as in Clarida, Gali and Gertler (1999).

solving the model. In a sense the problem is separable: first the Phillips curve is used in determining the optimal relationship between inflation and output; next, the aggregate demand relation can be used to back out an interest rate path. For analytical simplicity, we ignore this second step.

Nonetheless, analytical solutions are only available for a subset of the spectrum of loss functions examined in this paper.[12] Therefore, in Section 4.2 we use numerical solutions for a characterization of the remaining regimes. Prior to that we discuss analytical solutions for the four cases where such are available. Three of these have been derived previously elsewhere; the price level targeting case under discretion was solved in Vestin (2000) and the inflation targeting case with $j = 1$ under discretion and under commitment in CGG. The new results pertain to the case when the target is a two-period average (i.e. $j = 2$). We show in the following that delegating such a target to the central bank produces outcomes that are more efficient than the outcomes produced by adherence to an 'ordinary' one-period inflation target.

3.1 Analytic solutions

In this section we report optimal policy for (i) inflation targeting under commitment and three discretion cases: when the central bank is assigned (ii) a price level target, (iii) an inflation target where the inflation rate is calculated on a 'one-period' basis, and (iv) a two-period average inflation rate target. As already mentioned, the first three cases have been solved elsewhere but we reproduce the results here for ease of comparison. In each case we pay special attention to the resulting behavior of the output gap (which in this simple scaled-down model of the economy is the control variable), the price level and the (one-period) inflation rate, in particular their variances. We begin with the benchmark: inflation targeting under commitment.[13]

[12] A j-period moving average introduces $j - 1$ state variables. When $j > 2$ this becomes analytically intractable.

[13] In the analysis below we will assume that the goal for inflation is zero (i.e. $\pi^* = 0$) or that the goal for the price level is a constant one. These assumptions are made for analytical convenience, and have no effect on the results regarding the variances of inflation and output. The interpretation does however vary somewhat depending on the target values. In the case of $\pi^* = k > 0$, the variance of inflation should

(i) The "inflation-targeting-under-commitment" solution

As shown in CGG, solving (1) subject to (7) and (8) results in

$$\begin{aligned} p_t &= a^* \, p_{t-1} + b^* \, u_t, \\ \pi_t &= (a^* - 1) \, p_{t-1} + b^* \, u_t, \\ x_t &= -c^* \, p_{t-1} - d^* \, u_t, \end{aligned} \tag{9}$$

where the coefficients are defined by

$$\begin{aligned} a^* &= \frac{\left(\lambda \left(1+\beta\right) + \kappa^2\right) \left(1 - \sqrt{1 - 4\beta \left(\frac{\lambda}{\lambda(1+\beta)+\kappa^2}\right)^2}\right)}{2\lambda\beta}, \\ b^* &= \frac{a^*}{1 - a^* \beta\rho}, \\ c^* &= \frac{\left(1 - a^*\beta\right)\left(1 - a^*\right)}{\kappa}, \\ d^* &= \frac{1 - b^* \left[1 + \beta \left(1 - a^* \ - \rho\right)\right]}{\kappa}. \end{aligned}$$

Several features of the commitment solution should be noted. First, *the price level is stationary* (since, as shown in CGG , $0 \leq a^* \leq 1$, and only when the central bank has infinite concern for output stabilization is $a = 1$)[14]. Further, since c^* is strictly positive, optimal policy will be characterized by prolonged responses to one-time shocks (even when $\rho = 0$): as long as p_{t-1} remains above (below) trend, output will be kept below (above) trend. The coefficient $b^*(> 0)$ is increasing in λ, while $d^*(> 0)$ is decreasing in λ, reflecting that a greater concern for output stabilization will imply a more cautious response to shocks, the central bank allowing the shock to pass through to current inflation to a greater extent.

The variances of the output gap and the inflation rate are

$$\begin{aligned} \operatorname{var}\left\{\pi_t\right\}_{|\text{inflation target}} &= e^* \operatorname{var}\left\{u_t\right\}, \\ \operatorname{var}\left\{x_t\right\}_{|\text{inflation target}} &= f^* \operatorname{var}\left\{u_t\right\}, \end{aligned}$$

be interpreted as the variance of the deviation of inflation from this value. The results regarding levels, however, are of course affected, with constants being added in the case of non-zero target levels.

[14]Remember our assumption of $\pi^* = 0$. With a positive inflation target, the price level would instead be trend-stationary.

where

$$
\begin{aligned}
e^* &\equiv \frac{2b^{*2}(1-\rho)}{(1-a^*\rho)(1+a^*)}, \\
f^* &\equiv \frac{b^{*2}c^{*2}(1+a^*\rho)+d^*(1-a^{*2})\left[d^*(1-a^*\rho)+2\rho b^* c^*\right]}{(1-a^{*2})(1-a^*\rho)}.
\end{aligned}
$$

(ii) Price level targeting under discretion

Consider now the situation where the central bank is assigned a price level target, as in (4). In this case, Vestin (2000) shows that the output gap, the price level and inflation will evolve according to[15]

$$
\begin{aligned}
p_t &= \widetilde{a}\, p_{t-1} + \widetilde{b}\, u_t, \\
\pi_t &= (\widetilde{a}-1)\, p_{t-1} + \widetilde{b}\, u_t, \\
x_t &= -\widetilde{c}\, p_{t-1} - \widetilde{d}\, u_t,
\end{aligned}
\tag{10}
$$

where the coefficient $\widetilde{a}$ is defined by

$$
\begin{aligned}
\widetilde{a} &= \frac{\omega\widetilde{\lambda}}{\kappa^2+\omega^2\widetilde{\lambda}+\beta\widetilde{\lambda}(1-\omega\widetilde{a})}, \\
\omega &= 1+\beta(1-\widetilde{a}),
\end{aligned}
$$

($\widetilde{\lambda}$ being the relative weight on output stabilization versus *price level* stabilization) and the remaining coefficients $\widetilde{b}$, $\widetilde{c}$, and $\widetilde{d}$ are defined by

$$
\begin{aligned}
\widetilde{b} &= \frac{\omega\widetilde{\lambda}+\beta\rho\widetilde{\lambda}\left[2\omega\widetilde{b}-\left(1+\beta\rho\widetilde{b}\right)\right]}{\kappa^2+\omega^2\widetilde{\lambda}+\beta\widetilde{\lambda}(1-\omega\widetilde{a})}, \\
\widetilde{c} &= \frac{(1-\widetilde{a}\beta)(1-\widetilde{a})}{\kappa}, \\
\widetilde{d} &= \frac{1-\widetilde{b}\left[1+\beta(1-\widetilde{a}-\rho)\right]}{\kappa}.
\end{aligned}
$$

The important feature of this solution is its similarities with the "inflation targeting under commitment" solution. The coefficient $\widetilde{a}$ has the property that $0 \leq \widetilde{a}\left(\widetilde{\lambda}\right) < 1$, i.e. the price level is, naturally, stationary when the objective of monetary policy is to keep it stable. And as in the case of commitment above, the coefficient $\widetilde{c}$ is strictly positive,

[15] See also Appendix A for the complete derivation.

meaning that x_t will below (above) trend as long as p_{t-1} is above (below) it. Also, the coefficient $\widetilde{b}$ is increasing in $\widetilde{\lambda}$, while $\widetilde{d}$ is decreasing in $\widetilde{\lambda}$, with the same interpretation as above.

With a price level target the variances of the output gap and the inflation rate are

$$\begin{aligned} \operatorname{var}\{\pi_t\}_{|\text{price level target}} &= \widetilde{e} \operatorname{var}\{u_t\}, \\ \operatorname{var}\{x_t\}_{|\text{price level target}} &= \widetilde{f} \operatorname{var}\{u_t\}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{e} &\equiv \frac{2\widetilde{b}^2 (1-\rho)}{(1-\widetilde{a}\rho)(1+\widetilde{a})}, \\ \widetilde{f} &\equiv \frac{\widetilde{b}^2 \widetilde{c}^2 (1+\widetilde{a}\rho) + \widetilde{d}(1-\widetilde{a}^2)\left[\widetilde{d}(1-\widetilde{a}\rho) + 2\rho\widetilde{b}\widetilde{c}\right]}{(1-\widetilde{a}^2)(1-\widetilde{a}\rho)}. \end{aligned}$$

(iii) Inflation targeting under discretion: one-period inflation targeting ($j = 1$)

If the central bank is instructed to target the one-period inflation rate as in (2), CGG show that the output gap, the price level and the inflation rate will evolve according to

$$\begin{aligned} p_t &= p_{t-1} + \widehat{b}\, u_t, \\ \pi_t &= \widehat{b}\, u_t, \\ x_t &= -\widehat{d}\, u_t, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \widehat{b} &= \frac{\widehat{\lambda}}{\kappa^2 + \widehat{\lambda}(1-\beta\rho)}, \\ \widehat{d} &= \frac{\kappa}{\kappa^2 + \widehat{\lambda}(1-\beta\rho)} \quad \left(= \frac{\kappa}{\widehat{\lambda}}\widehat{b}\right), \end{aligned}$$

($\widehat{\lambda}$ being the relative weight on output stabilization versus *inflation* stabilization). The major difference compared to the two previous cases is that $\widehat{a} = 1$ and $\widehat{c} = 0$. This means that the price level is no longer stationary; indeed, if $\rho = 0$ it would be a random walk. Further, $\widehat{c} = 0$ implies one-time responses to one-time shocks, or, differently put, policy responses will be persistent only if the shocks themselves are persistent.

In this case when $j = 1$, with inflation and output being functions of the exogenous shocks u_t only, the variances of the output gap and the inflation rate are particularly straightforward to derive:

$$\begin{aligned} \operatorname{var}\{\pi_t\}_{|\text{inflation target}} &= \widehat{b}^2 \operatorname{var}\{u_t\}, \\ \operatorname{var}\{x_t\}_{|\text{inflation target}} &= \widehat{d}^2 \operatorname{var}\{u_t\}. \end{aligned}$$

(iv) Inflation targeting under discretion: two-period moving average ($j = 2$)

We now come to the case when the central bank is instructed to target the two period average inflation rate (thus $j = 2$ in (6)). This case has not been solved earlier, and the complete derivation can be found in Appendix A. There it is shown that the price level, inflation and the output gap will evolve according to

$$\begin{aligned} p_t &= \overline{a}\, p_{t-1} - (\overline{a} - 1)\, p_{t-2} + \overline{b}\, u_t, \\ \pi_t &= (\overline{a} - 1)\; p_{t-1} - (\overline{a} - 1)\; p_{t-2} + \overline{b}\, u_t, \\ x_t &= -\overline{c}\, p_{t-1} + \overline{c}\, p_{t-2} - \overline{d}\, u_t, \end{aligned} \tag{12}$$

where $\overline{a}$ and $\overline{b}$ are determined simultaneously by

$$\begin{aligned} \overline{a} &= 1 - \frac{\kappa^2}{\kappa^2 (1 + \beta\overline{a}) + 4\overline{\lambda} (1 - \beta (\overline{a} - 1))^2}, \\ \overline{b} &= \frac{4\overline{\lambda} (1 - \beta (\overline{a} - 1)) \left(1 + \beta\rho\overline{b}\right) - \beta\rho\kappa^2\overline{b}}{\kappa^2 (1 + \beta\overline{a}) + 4\overline{\lambda} (1 - \beta (\overline{a} - 1))^2}, \end{aligned}$$

and

$$\begin{aligned} \overline{c} &\equiv \frac{(\overline{a} - 1)(\overline{a}\beta - \beta - 1)}{\kappa}, \\ \overline{d} &\equiv \frac{1 - \overline{b}\,(1 + \beta (1 - \overline{a} - \rho))}{\kappa}, \end{aligned}$$

where $\overline{\lambda}$ is the relative weight on output stabilization versus *two-year-average inflation* stabilization.

The general form of the equations for inflation and output have changed somewhat compared to the earlier cases, e.g. p_{t-2} also appears. Now, with a goal formulated in terms of a two-period average, it should not be surprising that variables dated two periods ago affect the conduct of optimal monetary policy. A fundamental consequence of the two-period average is that *bygones are no longer bygones.* Consideration must be taken to what 'one-period' inflation was last period — if it was below the target π^*, this period's inflation must be above π^* in order to keep the average 'close' to π^* (i.e. with due consideration taken to output concerns). And, of course, lagged inflation affecting policy is the same as the price level lagged two periods affecting policy.

The following can be said about the properties of the coefficients. First, $0 \leq \overline{a} < 1$, the consequence being that (one-period) inflation will for a time oscillate around zero following a temporary shock.[16] The reason why this is optimal is, as was just explained, that it will result in the two-year average being 'close' to target (the target being normalized to 0). Further, the coefficient $\overline{b} \geq 0$ and is increasing in $\overline{\lambda}$, while $\overline{d} \geq 0$ and is decreasing in $\overline{\lambda}$ with the same interpretion as before - the greater the concern for real variability the more will a given shock be allowed to feed through to current inflation. Finally, the coefficient $\overline{c} \geq 0$ and is decreasing in $\overline{\lambda}$. This means that with a goal for the two-year average inflation rate, monetary policy will be characterized by persistence, even following temporary shocks. This feature of the solution is a central one, since it means that the solution exhibits history dependence.

The variances of the output gap and the inflation rate are

$$\begin{aligned} \text{var}\left\{\pi_t\right\}_{|\text{inflation target, j=2}} &= \overline{e}\ \text{var}\left\{u_t\right\}, \\ \text{var}\left\{x_t\right\}_{|\text{inflation target, j=2}} &= \overline{f}\ \text{var}\left\{u_t\right\}, \end{aligned}$$

where

$$\begin{aligned} \overline{e} &\equiv \frac{\overline{b}^2\left(1+\overline{a}\rho-\rho\right)}{\left(1-\overline{a}\rho+\rho\right)\overline{a}\left(2-\overline{a}\right)}, \\ \overline{f} &\equiv \frac{\overline{b}^2\overline{c}^2\ \left(1+\overline{a}\rho-\rho\right)+\overline{a}\left(2-\overline{a}\right)\overline{d}\left[\overline{d}\left(1-\overline{a}\rho+\rho\right)+2\overline{b}\overline{c}\rho\right]}{\left(1-\overline{a}\rho+\rho\right)\overline{a}\left(2-\overline{a}\right)}. \end{aligned}$$

Comparison of regimes

Using the analytical solutions for the variances of output and inflation under the four regimes we have studied so far, we can compare the monetary-policy trade-offs faced in

[16] Numerical simulations reveal that these oscillations only arise for very small values of λ.

each case. This is done by constructing 'variance frontiers', i.e. combinations of var $\{\pi_t\}$ and var $\{x_t\}$ for different values of the preference parameters $\lambda, \widetilde{\lambda}, \widehat{\lambda}$,and $\overline{\lambda}$. Such frontiers are also called 'efficient policy frontiers' since points outside the frontier are inefficient, while points inside are infeasible. Figure 1 contains the variance frontiers for the four cases. When plotting these frontiers we have assumed that $\beta = 0.96$, $\kappa = 0.2$, and $\rho = 0.5$.[17] The commitment frontier (the thin dashed line) lies closest to the origin, and is thus the most favorable one.[18] The "$j = 1$ under discretion" frontier (thick dashed line) is farthest to the north-east, and thus the least favorable. The price level targeting frontier (thick dotted line) almost coincides with the commitment frontier as was shown in Vestin (2000).[19] The new result here pertains to the $j = 2$ case (the thin solid line). What we see in Figure 1 is that it constitutes an intermediate case, i.e. *with a target for two-period average inflation a better trade-off is attained than with a one-period inflation target.*

[Figure 1 about here:
Variance frontiers based on analytical expressions for $var\,\{\pi_t\}$ and $var\,\{x_t\}$.]

To explain the intuition behind this result, we note that inflation expectations, when the two-period average inflation rate is targeted, are given by

$$\pi_{t+1|t} = (\overline{a} - 1)\,\pi_t + \overline{b}\,u_{t+1|t}. \tag{13}$$

Since $0 \leq \overline{a} \leq 1$, an inflation rate above target (the target being normalized to zero) leads to expectations of inflation being lower than target in the subsequent period, i.e. *following a positive shock the two-year average target automatically generates expectations of deflation.* By inspecting the Phillips relation (7), it follows that the instrument x_t does not have to be depressed as much since inflation expectations simultaneously fall. It is in this sense that the monetary policy maker is helped by suitably changing expectations, the result being an improved short-run monetary-policy trade-off compared with the inflation

[17] We think of this model as being an annual one. Setting $\beta = 0.96$ corresponds to a quarterly discount rate of 0.99. The value for κ is a simple average of the estimates obtained by Rotemberg and Woodford (1997) and Roberts (1995), respectively. Finally, the value for ρ was simply assumed.

[18] See Appendix C for a further discussion on this.

[19] For values of ρ close to 1, the difference becomes discernable to the eye.

targeting ($j = 1$) case.[20] This improved short-run trade-off in turn means that the variance frontier lies closer to the origin, and closer to the commitment solution.

To end this section, a few words should be said about the robustness of the results shown in Figure 1 with respect to the parameter values used. Changing the values of κ and/or ρ does not alter the qualitative results. Larger values of κ increases var $\{x_t\}$, while larger values of ρ increases both var $\{\pi_t\}$ and var $\{x_t\}$, but the ordering of the frontiers — the price level target frontier being closest to the commitment solution, the $j = 1$ frontier farthest from it, and the two-year average frontier lying in between — does not change. The results are also robust for all reasonable values of β, i.e. remembering that it is a discount factor.

Impulse responses - the 'stabilization bias'

Another way of comparing the four regimes is with respect to their dynamic responses to temporary shocks. As discussed in e.g. Woodford (1999b), the discretionary implementation of an inflation target will lead to suboptimal dynamic responses, the so-called 'stabilization bias'.[21] Here we examine how the discretionary dynamic responses change — with a caveat, soon to be explained — when different 'modified' objective functions are assigned to the central bank. Figure 2 contains the dynamic responses of inflation and output to a temporary shock, i.e. here we assume that $\rho = 0$.[22] Further, we set the preference parameter equal to 0.2. However, (and this is the caveat) remember that the preference parameters do not apply to the same comparisons (e.g. λ is the preference parameter in relation to the choice between output variability and inflation variability,

[20] By equation (11) inflation expectations when $j = 1$ is simply $\pi_{t+1|t} = \widehat{b}\, u_{t+1|t}$. Note that 'suitably' adapting expectations also materialize in the case of a price level target. By (10) it follows that inflation expectations when the central bank targets the price level are given by

$$\pi_{t+1|t} = (\widetilde{a} - 1)\, p_t + \widetilde{b}\, u_{t+1|t}.$$

Since $0 \leq \widetilde{a} < 1$, a positive value of p_t (i.e. the price level is above target, the latter being normalized to zero) will have a dampening effect on inflation expectations.

[21] For early references on this issue, see Backus and Driffil (1986) and Currie and Levine (1993). See also Söderlind (1999).

[22] We continue to assume that $\beta = 0.96$ and $\kappa = 0.2$.

while $\overline{\lambda}$ concerns the choice between output variability and variability in the two-year average inflation rate). Thus the curves cannot be compared directly, e.g. absolute magnitudes cannot be compared. But we can examine the graphs to learn something about the general characteristics of the dynamic responses.

Consider first the response under commitment (thin dashed line): even though the shock is temporary, the policy response is persistent. The reason why this is so is that it produces deflation in the periods following the shock, expectations of which are desireable since they imply that the response today can milder (and the overall loss smaller) (see Woodford, 1999). In contrast, inflation targeting under discretion gives a one-time (and 'large') response to a one-time shock, and inflation is back to its pre-shock value after one period. Consider now the two cases with 'modified' objective functions. With a price level target (dotted line) the response displays the same characteristic as the commitment case, i.e. of a persistent response, and of 'undershooting' of inflation. The same applies to the two-period average case, i.e. the output response is persistent (even when the shock is not) and there is deflation after the shock (however only for one period).

[Figures 2 and 3 about here:

Dynamic responses of inflation and output to a temporary shock.]

3.2 Numerical solutions

As we have noted previously, we are unable to provide analytical solutions for $j > 2$. Instead we rely on numerical solutions for a characterization of these regimes. Appendix B shows how the simple model used in this section is written in state-space form. The parameter values that we have used are the same as those that were used above (when discussing the analytical solutions), i.e. $\beta = 0.96, \rho = 0.5$ and $\kappa = 0.2$. We have in mind primarily an annual model, which means that our parameter values are comparable to those used in Woodford (1999b) and McCallum & Nelson (2000).

Variance frontiers

In Figure 4, variance frontiers are shown for four cases: $j = 4$ and $j = 16$, together with the price level targeting case and $j = 1$ case for comparison.[23] The figure shows that as the window over which the average inflation rate is calculated increases, the corresponding variance frontier moves closer to the origin — increasing j improves the monetary-policy trade-off.

[Figure 4 about here:
Variance frontiers based on numerical solutions for $var\{\pi_t\}$ and $var\{x_t\}$.]

When $j = 16$, the frontier almost coincides with the price level targeting frontier (which in this simple model almost coincides with the commitment frontier). We mentioned in the Introduction that our analysis of average inflation targeting can be said to explore the intermediate regimes that lie between inflation targeting on the one hand, and price level targeting on the other, and we argued that as j grows we approach the price level targeting case. From Figure 4 we can get a rough estimate of how large j must be in order for average inflation targeting to generate the same monetary policy trade-off as price level targeting. In the particular model that employ in this section, that value of j appears to be around 16.

Impulse responses

Figures 5 and 6 show the impulse responses obtained from the numerical solutions for the same four regimes as in Figure 4; the caveat regarding the direct comparison of these curves mentioned in conjunction with Figures 2 and 3 also applies here. In Figure 6 we can see that the policy responses to a temporary shock[24] are persistent also in the case of 'higher' j. Figure 5 reveals that deflation arises in the periods following the shock, and that inflation is back on its pre-shock value after three periods when $j = 4$ and after fifteen periods when $j = 16$.

[23]The latter two frontiers of course coincide with the corresponding curves in Figure 1. We only show two cases of 'higher' j, so as to not clutter the diagram.

[24]Remember that $\rho = 0$ in the impulse responses.

[Figures 5 and 6 about here:

Dynamic responses of inflation and output to a temporary shock - numerical solution.]

4 A Hybrid Model

An important message from the analysis of the previous section is that with a forward-looking Phillips curve, the policy trade-off will improve if inflation expectations can be made to adjust in a suitable way. Earlier studies have shown that this can be accomplished with a price level target. The new result in this paper is that an average inflation target will also cause inflation expectations to change in a desireable direction, i.e. in such a way that the monetary-policy trade-off is improved.

A natural question to ask now is how sensitive these results are to the particulur form of the Phillips curve. The results above were obtained in a purely forward-looking specification, where lagged inflation rates play no role in the determination of current inflation. Such specifications have been criticized on the grounds of poor empirical performance, and it has been argued that lagged inflation rates must also enter the Phillips curve in order to properly account for actual inflation (see e.g. Fuhrer, 1997). Such 'hybrid' formulations, containing both expectations of future levels of inflation and lagged inflation, can be obtained theoretically in a number of ways. Fuhrer and Moore (1995) do so by assuming that relative real wages are set in a staggered fashion. In derivations based on optimizing behavior, Hallsten (1999) obtains the hybrid formulation in a model with quadratic price adjustment costs (thus extending results obtained by Rotemberg, 1982), while Amato and Laubach (2000) obtain a hybrid Phillips curve by introducing habit formation in consumption in a general equilibrium model with price stickiness à la Calvo (1983).

Thus, consider a slightly richer model of the economy than in Section 3. It is still a one-equation model (with an additional equation for exogenous persistence) but one where we allow for both forward- and backward-looking behavior in the determination of current inflation:

$$\pi_t = (1-\alpha)\,\pi_{t-1} + \alpha\beta\pi_{t+1|t} + \kappa x_t + u_t, \tag{14}$$
$$u_t = \rho u_{t-1} + \varepsilon_t.$$

For simplicity we continue to assume, as in Section 3, that the output gap, x_t, is the control variable. Also, as before, exogenous persistence is introduced via the autoregressive process u_t. The forward- and backward looking elements of the Phillips curve enter with weights α and $(1-\alpha)$, respectively.

When $\alpha = 1$ we have exactly the model of Section 3, and there is no need to reiterate the results here. Consider instead what happens to the variance frontiers when the weight on expected inflation (the 'forward-looking component' of the Phillips curve), α, falls.[25] Intuitively, as α falls there is less room for monetary policy being 'helped' by suitably adjusting expectations, since expectations play a smaller role in the determination of current inflation. Therefore one could easily suspect that price level targeting, e.g., should not produce as favorable outcomes as with a purely forward-looking Phillips curve. And this is also what happens. Consider Figure 7 which shows the variance frontiers for four values of α.[26] In the top left-hand panel the degree of 'forward-lookingness' is still high ($\alpha = 0.8$), and the ordering of the frontiers remains the same as when the Phillips curve is entirely forward-looking; thus the $j = 1$ variance frontier is the least favorable, $j = 2$ generates a variance frontier that lies closer to the origin[27], and the price level targeting frontier is closest to the commitment case.[28] In the remaining panels the value of α

[25] Empirical estimates of α include Fuhrer (1997) who could not reject that $\alpha = 0$ (although he argues that a model with some forward-looking behavior has more realistic dynamics), and Rudebusch (2000) who estimates an equation like (14) for the U.S. using quarterly data 1968:III to 1996:IV, and obtained the estimate $\alpha = 0.29$.

[26] The model has been solved numerically, assuming that $\{\beta,\ \kappa,\ \rho\} = \{0.96,\ 0.2,\ 0.5\}$.

[27] In Figure 7 we only show $j = 2$, but in separate graphs (not reported in this version of the paper) higher values of j were tried, and they generated frontiers that move closer to the origin as j was increased.

[28] We are now implicitly using the same welfare criterion as in Section 3. It has recently been brought to our attention that Steinsson (2000) presents a model with explicit microfoundations which results in a Phillips curve with inflation persistence. He also derives the correct social welfare function in such an economy. Ideally, one should use such a welfare measure in this section when comparing the regimes, and we will explore this possibility in future work.

falls, and the ordering of the frontiers changes. Generally speaking, as the importance of expected inflation in the determination of current inflation diminishes, price level targeting provides less favorable outcomes (i.e. the price level frontier begins to move out). Furthermore, the $j = 1$ and $j = 2$ frontiers move closer to each other, eventually almost coinciding for small values of α. But, as demonstrated in the bottom left-hand panel of Figure 7 (corresponding to $\alpha = 0.4$) there are values of α when the $j = 2$ frontier lies closest to the origin. Thus, Figure 9 shows that *there is a range of values for α where a two-period average target provides more efficient outcomes than does both a price level target and a one-period inflation target.*

[Figure 7 about here:
Variance frontiers for hybrid model (numerical solutions).]

Thus, the relative merits of price level targeting, inflation targeting and average inflation targeting depend critically on the importance of forward-looking behavior in the Phillips curve. In purely forward-looking specifications of the Phillips curve, price level targeting provides the most favorable outcome. But as the importance of expected inflation diminishes, price level targeting begins to perform more poorly, with inflation targeting and average inflation targeting starting to provide the better outcomes. For intermediate values of α average inflation targeting dominates one-period inflation targeting, while for very small values of α there is hardly any difference.

5 Conclusions

In "Optimal Monetary Policy Inertia", Michael Woodford (1999a) shows that optimal monetary policy *is* inertial. In other words, optimal policy responses — and also inflation and output gaps — are characterized by persistence, even following purely temporary shocks. These optimal solutions, obtained under commitment, are thus said to exhibit history dependence.

This paper belongs to a line of research that asks what can be done when commitment is not possible. The maintained assumption throughout the paper is that the social optimum is given by inflation targeting under commitment. With this in mind, we ask how society should design the mandate delegated to, and implemented (in a discretionary fashion) by an independent central bank. A recent paper by Vestin (2000) suggests that, in a purely forward-looking model, a price level target will provide more efficient outcomes than an inflation target. In this paper we also examine another set of policies, *average* inflation targeting, whereby the central bank is instructed to minimize a quadratic function in the output gap and average inflation measured over j years.

The results of this analysis show that when the Phillips curve has forward-looking components, a goal for average inflation will provide better outcomes than a 'conventional' one-period inflation target in the sense of producing variance frontiers that lie closer to the origin. The basic intuition behind this result is that average inflation targeting introduces history dependence in expectations. For example, with a target for two-period average inflation, a positive shock to inflation in one period will lead to expectations of lower-than-target inflation in the following period. And when the Phillips-curve is forward-looking, this change in expectations will improve the short-run trade-off faced by the monetary policymaker.

In purely forward-looking models (or more specifically models in which lagged inflation rates have no role in the determination of current inflation) average inflation targeting is dominated by price level targeting. The price level targeting variance frontier lies strictly inside the frontiers generated by average inflation targeting, even though the latter move towards the price level targeting frontier as j increases . But we also demonstrate that in models where the Phillips curve has both forward- and backward-looking components, there are cases when the average inflation target provides more efficient outcomes than both 'ordinary' inflation targeting and price level targeting.

Thus, the relative merits of price level targeting, inflation targeting and average inflation targeting depend critically on the structural parameters of the economy in general, and more specifically on the degree of persistence in the Phillips curve. Needless to say, theoretical and empirical work on these factors remains an important task.

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A Analytical solutions to the simple model

Price level targeting

As shown in Vestin (2000) the model is solved by setting up the value function

$$\begin{aligned} V(p_{t-1}, u_t) &= E_t \left\{ \min_{x_t} \left[\frac{1}{2} \left({p_t}^2 + \tilde{\lambda} x_t^2 \right) + \beta V(p_t, u_{t+1}) \right] \right\} \\ \text{s.t.} \quad \pi_t &= \beta \pi_{t+1|t} + \kappa x_t + u_t, \\ u_t &= \rho u_{t-1} + \varepsilon_t. \end{aligned} \tag{A.1}$$

Using the definition $\pi_t \equiv p_t - p_{t-1},$ the Phillips curve may be rewritten:

$$p_t - p_{t-1} = \beta \left(p_{t+1|t} - p_t \right) + \kappa x_t + u_t,$$

which solving for x_t gives

$$x_t = \frac{1}{\kappa} (1 + \beta) p_t - \frac{\beta}{\kappa} p_{t+1|t} - \frac{1}{\kappa} (p_{t-1} + u_t). \tag{A.2}$$

The linear-quadratice structure of the problem ensures that the state variable p_t will follow

$$p_{t+1} = a_{t+1} p_t + b_{t+1} u_{t+1}, \tag{A.3}$$

(where the coefficients a_{t+1} and b_{t+1} remain to be determined) implying that

$$p_{t+1|t} = a_{t+1} p_t + b_{t+1} \rho u_t. \tag{A.4}$$

Thus equation (A.2) may be written

$$\begin{aligned} x_t &= \frac{1}{\kappa} (1 + \beta) p_t - \frac{\beta}{\kappa} (a_{t+1} p_t + b_{t+1} \rho u_t) - \frac{1}{\kappa} (p_{t-1} + u_t), \\ &= -\frac{1}{\kappa} p_{t-1} + \frac{1}{\kappa} [1 + \beta (1 - a_{t+1})] \; p_t - \frac{1}{\kappa} [1 + \beta \rho b_{t+1}] u_t, \end{aligned} \tag{A.5}$$

or, solving for p_t

$$p_t = \frac{\kappa}{1 + \beta (1 - a_{t+1})} x_t + \frac{1}{1 + \beta (1 - a_{t+1})} p_{t-1} + \frac{1 + \beta \rho b_{t+1}}{1 + \beta (1 - a_{t+1})} u_t.$$

Thus

$$\frac{\partial p_t}{\partial x_t} = \frac{\kappa}{1 + \beta (1 - a_{t+1})}.$$

The first order condition for the problem is:

$$0 = E_t \left\{ \frac{\partial p_t}{\partial x_t} p_t + \lambda x_t + \beta \frac{\partial V_{t+1}}{\partial p_t} \frac{\partial p_t}{\partial x_t} \right\}.$$

Here we need a guess for the value function. It is:

$$\begin{aligned} V(p_{t-1}, u_t) &= \gamma_{0,t} + \gamma_{1,t} p_{t-1} + \frac{1}{2}\gamma_{2,t} p_{t-1}^2 + \gamma_{3,t} p_{t-1} u_t + \gamma_{4,t} u_t + \frac{1}{2}\gamma_{5,t} u_t^2 \\ &\Longrightarrow \\ E_t \frac{\partial V_{t+1}}{\partial p_t} &= \gamma_{1,t+1} + \gamma_{2,t+1} p_t + \gamma_{3,t+1} \rho u_t. \end{aligned}$$

Thus we may proceed with the first-order condition:

$$\begin{aligned} 0 &= \frac{\kappa}{1+\beta(1-a_{t+1})} p_t + \lambda x_t + \beta \left(\gamma_{1,t+1} + \gamma_{2,t+1} p_t + \gamma_{3,t+1} \rho u_t \right) \left(\frac{\kappa}{1+\beta(1-a_{t+1})} \right), \\ 0 &= \frac{\kappa}{1+\beta(1-a_{t+1})} p_t + \frac{\lambda}{\kappa} \left([1+\beta(1-a_{t+1})]\, p_t - p_{t-1} - [1+\beta\rho b_{t+1}] \right) u_t \\ &\quad + \beta \left(\gamma_{1,t+1} + \gamma_{2,t+1} p_t + \gamma_{3,t+1} \rho u_t \right) \left(\frac{\kappa}{1+\beta(1-a_{t+1})} \right), \end{aligned}$$

and

$$\begin{aligned} \left(\frac{\kappa^2 + \lambda [1+\beta(1-a_{t+1})]^2 + \kappa^2 \beta \gamma_{2,t+1}}{\kappa [1+\beta(1-a_{t+1})]} \right) p_t &= -\frac{\kappa \beta \gamma_{1,t+1}}{1+\beta(1-a_{t+1})} + \lambda \frac{1}{\kappa} p_{t-1} \\ &\quad + \left[\frac{\lambda [1+\beta(1-a_{t+1})] [1+\beta\rho b_{t+1}] - \beta \gamma_{3,t+1} \rho \kappa^2}{\kappa [1+\beta(1-a_{t+1})]} \right] u_t, \end{aligned}$$

which solving for p_t gives

$$\begin{aligned} p_t &= -\frac{\kappa^2 \beta \gamma_{1,t+1}}{\kappa^2 + \lambda [1+\beta(1-a_{t+1})]^2 + \kappa^2 \beta \gamma_{2,t+1}} + \frac{\lambda [1+\beta(1-a_{t+1})]}{\kappa^2 + \lambda [1+\beta(1-a_{t+1})]^2 + \kappa^2 \beta \gamma_{2,t+1}} p_{t-1} \\ &\quad + \frac{\lambda [1+\beta(1-a_{t+1})] [1+\beta\rho b_{t+1}] - \beta \gamma_{3,t+1} \rho \kappa^2}{\kappa^2 + \lambda [1+\beta(1-a_{t+1})]^2 + \kappa^2 \beta \gamma_{2,t+1}} u_t, \\ &= \frac{\lambda [1+\beta(1-a_{t+1})]}{\kappa^2 + \lambda [1+\beta(1-a_{t+1})]^2 + \kappa^2 \beta \gamma_{2,t+1}} p_{t-1} + \frac{\lambda [1+\beta(1-a_{t+1})] [1+\beta\rho b_{t+1}] - \beta \gamma_{3,t+1} \rho \kappa^2}{\kappa^2 + \lambda [1+\beta(1-a_{t+1})]^2 + \kappa^2 \beta \gamma_{2,t+1}} u_t, \end{aligned}$$

The envelope theorem applied to (A.1) gives

$$\begin{aligned} \gamma_{2,t} &= \frac{\lambda}{\kappa^2} \left\{ 1 - [1+\beta(1-a_{t+1})]\, a_t \right\}, \\ \gamma_{3,t} &= \frac{\lambda}{\kappa^2} \left\{ (1+\beta\rho b_{t+1}) - [1+\beta(1-a_{t+1})]\, b_t \right\}. \end{aligned}$$

Substituting these expressions into the equation for p_t above, and comparing with (A.3) imply the following equations:

$$
\begin{aligned}
a_t &= \tfrac{\lambda[1+\beta(1-a_{t+1})]}{\kappa^2+\lambda[1+\beta(1-a_{t+1})]^2+\beta\lambda\{1-[1+\beta(1-a_{t+2})]a_{t+1}\}}, \\
b_t &= \tfrac{\lambda[1+\beta(1-a_{t+1})]+\beta\lambda\rho\{[1+\beta(1-a_{t+1})]b_{t+1}-(1+\beta\rho b_{t+1})+[1+\beta(1-a_{t+2})]b_{t+1}\}}{\kappa^2+\lambda[1+\beta(1-a_{t+1})]^2+\beta\lambda\{1-[1+\beta(1-a_{t+2})]a_{t+1}\}}.
\end{aligned} \tag{A.6}
$$

Thus the solution for p_t is

$$p_t = ap_{t-1} + bu_t, \tag{A.7}$$

where a and b are the stationary solutions to (A.6).

To get an expression for x_t, substitute these stationary solution for the equations above into (A.5) to obtain:

$$
\begin{aligned}
x_t &= -\frac{1}{\kappa}p_{t-1} + \frac{1}{\kappa}\left[1+\beta\left(1-a\right)\right]\, p_t - \frac{1}{\kappa}\left[1+\beta\rho b\right] u_t \\
&= -\frac{1}{\kappa}p_{t-1} + \frac{1}{\kappa}\left[1+\beta\left(1-a\right)\right]\left(a\, p_{t-1} + bu_t\right) - \frac{1}{\kappa}\left[1+\beta\rho b\right] u_t \\
&= \frac{1}{\kappa}\left[a + a\beta - a^2\beta - 1\right]\, p_{t-1} + \frac{1}{\kappa}\left[b + \beta b - ab\beta - 1 - \beta\rho b\right] u_t \\
&= -\frac{1}{\kappa}\left[1 - a - a\beta + a^2\beta\right]\, p_{t-1} - \frac{1}{\kappa}\left[1 - b - \beta b + ab\beta + \beta\rho b\right] u_t \\
&= -\frac{1}{\kappa}\left(1-a\right)\left(1-a\beta\right)\, p_{t-1} - \frac{1}{\kappa}\left[1 - b\left(1+\beta\left(1-a-\rho\right)\right)\right] u_t,
\end{aligned}
$$

or

$$x_t = -c\, p_{t-1} - du_t. \tag{A.8}$$

Variances

The variance for is π_t obtained first by noting that, by (A.7)

$$\pi_t = \left(a-1\right) p_{t-1} + bu_t.$$

Thus

$$\text{var}\left\{\pi_t\right\} = \left(a-1\right)^2 \text{var}\left\{p_{t-1}\right\} + b^2\text{var}\left\{u_t\right\} + 2\left(a-1\right) b \text{ cov}\left\{p_t, u_t\right\}. \tag{A.9}$$

Since, by (A.7)

$$\text{var}\left\{p_t\right\} = a^2\text{var}\left\{p_{t-1}\right\} + b^2\text{var}\left\{u_t\right\} + 2ab \text{ cov}\left\{p_{t-1}, u_t\right\},$$

and

$$\begin{aligned}
\operatorname{cov}\{p_{t-1}, u_t\} &= \operatorname{cov}\{ap_{t-2} + bu_{t-1}, \rho u_{t-1} + \varepsilon_t\} \\
&= a\rho \operatorname{cov}\{p_{t-2}, u_{t-1}\} + b\rho \operatorname{var}\{u_t\} \\
&\Downarrow \\
\operatorname{cov}\{p_{t-1}, u_t\} &= \frac{b\rho}{1-a\rho}\operatorname{var}\{u_t\},
\end{aligned}$$

(due to stationarity) we can write

$$\begin{aligned}
\operatorname{var}\{p_t\} &= a^2\operatorname{var}\{p_{t-1}\} + b^2\operatorname{var}\{u_t\} + 2ab\frac{b\rho}{1-a\rho}\operatorname{var}\{u_t\} \\
&\Downarrow \\
\operatorname{var}\{p_t\} &= \frac{b^2(1+a\rho)}{(1-a^2)(1-a\rho)}\operatorname{var}\{u_t\},
\end{aligned}$$

Substituting into(A.9) gives

$$\begin{aligned}
\operatorname{var}\{\pi_t\} &= \left[(a-1)^2\frac{b^2(1+a\rho)}{(1-a^2)(1-a\rho)} + b^2 + 2(a-1)b\frac{b\rho}{1-a\rho}\right]\operatorname{var}\{u_t\} \\
&= \left[\frac{(1-a)^2 b^2(1+a\rho)}{(1-a^2)(1-a\rho)} + b^2 - \frac{2(1-a)b^2\rho}{1-a\rho}\right]\operatorname{var}\{u_t\} \\
&= b^2\frac{(1-a)}{(1-a\rho)}\left[\frac{(1+a\rho)}{(1+a)} + \frac{(1-a\rho)}{(1-a)} - 2\rho\right]\operatorname{var}\{u_t\} \\
&= \frac{2(1-\rho)}{(1-a\rho)(1+a)}b^2\operatorname{var}\{u_t\}.
\end{aligned}$$

Similarly, the variance of output is obtainen from (A.8)

$$\operatorname{var}\{x_t\} = c^2\operatorname{var}\{p_t\} + d^2\operatorname{var}\{u_t\} + 2cd\operatorname{cov}\{p_{t-1}, u_t\}.$$

Substituting in the appropriate expressions we get

$$\begin{aligned}
\operatorname{var}\{x_t\} &= \left[\frac{b^2c^2(1+a\rho)}{(1-a^2)(1-a\rho)} + d^2 + \frac{2bcd\rho}{1-a\rho}\right]\operatorname{var}\{u_t\}, \\
&= \left[\frac{b^2c^2(1+a\rho) + d(1-a^2)[d(1-a\rho) + 2bc\rho]}{(1-a^2)(1-a\rho)}\right]\operatorname{var}\{u_t\}.
\end{aligned}$$

One period target

In this case the value function may be written

$$\begin{aligned}
V(u_t) &= E_t\left\{\min_{x_t}\left[\frac{1}{2}\left({\pi_t}^2 + \lambda x_t^2\right) + \beta V(u_{t+1})\right]\right\}, \\
\text{s.t.}\quad \pi_t &= \beta\pi_{t+1|t} + \kappa x_t + u_t.
\end{aligned}$$

See Clarida, Gali and Gertler (1999).

Two period average target

The analytical solution for the case of a two-period average in the loss function is obtained in the following manner is similar to the price level targeting case. The value function is

$$
\begin{aligned}
V_t(\pi_{t-1}, u_t) &= \min_{x_t} \frac{1}{2}\left[\left(\frac{\pi_t + \pi_{t-1}}{2}\right)^2 + \lambda x_t^2\right] + \beta \mathrm{E}_t\left[V_{t+1}(\pi_t, u_{t+1})\right], \\
\text{s.t. } \pi_t &= \beta \pi_{t+1|t} + \kappa x_t + u_t. \qquad \text{(A.10)}
\end{aligned}
$$

The state variable will follow a linear path (which follows from the fact that a quadratic loss function implies that the control variable will be a linear function of the state variables), i.e. $\pi_{t+1} = a_{t+1}\pi_t + b_{t+1}u_{t+1}$, which implies

$$\pi_{t+1|t} = a_{t+1}\pi_t + b_{t+1}\rho u_t. \qquad \text{(A.11)}$$

Substitute (A.11) into (A.10) and solve for x_t:

$$x_t = \frac{1 - \beta a_{t+1}}{\kappa}\pi_t - \frac{1 + \beta\rho b_{t+1}}{\kappa}u_t. \qquad \text{(A.12)}$$

Solving for π_t we obtain

$$\pi_t = \frac{\kappa}{1 - \beta a_{t+1}}x_t + \frac{1 + \beta\rho b_{t+1}}{1 - \beta a_{t+1}}u_t. \qquad \text{(A.13)}$$

The first order condition for the optimization problem is:

$$\left(\frac{\pi_t + \pi_{t-1}}{2}\right)\frac{1}{2}\frac{\partial \pi_t}{\partial x_t} + \lambda x_t + \beta \mathrm{E}_t\left[\frac{\partial V_{t+1}(\pi_t, u_{t+1})}{\partial \pi_t}\frac{\partial \pi_t}{\partial x_t}\right] = 0.$$

In order to proceed we guess a value function:

$$V_t(\pi_{t-1}, u_t) = \gamma_{0,t} + \gamma_{1,t}\pi_{t-1} + \frac{1}{2}\gamma_{2,t}\pi_{t-1}^2 + \gamma_{3,t}\pi_{t-1}u_t + \gamma_{4,t}u_t + \frac{1}{2}\gamma_{5,t}u_t^2.$$

Thus

$$
\begin{aligned}
\mathrm{E}_t\left[\frac{\partial}{\partial \pi_t}V_{t+1}(\pi_t, u_{t+1})\right] &= \mathrm{E}_t\left[\gamma_{1,t+1} + \gamma_{2,t+1}\pi_t + \gamma_{3,t+1}u_{t+1}\right] \\
&= \gamma_{1,t+1} + \gamma_{2,t+1}\pi_t + \gamma_{3,t+1}\rho u_t,
\end{aligned}
$$

meaning that $\gamma_{0,t}$, $\gamma_{4,t}$ and $\gamma_{5,t}$ will be of no interest. Furthermore, $\gamma_{1,t+1}$ concerns only the level of inflation, why we set it to zero. Continuing with the first-order conditions

$$\begin{aligned}
\left[\frac{\pi_t}{4} + \frac{\pi_{t-1}}{4} + \beta\left(\gamma_{2,t+1}\pi_t + \gamma_{3,t+1}\rho u_t\right)\right]\frac{\partial \pi_t}{\partial x_t} + \lambda x_t &= 0, \\
\left[\left(\frac{1}{4} + \beta\gamma_{2,t+1}\right)\pi_t + \frac{1}{4}\pi_{t-1} + \beta\gamma_{3,t+1}\rho u_t\right]\frac{\kappa}{1-\beta a_{t+1}} + \lambda x_t &= 0,
\end{aligned}$$

where the last line follows from (A.13). Inserting (A.12) into the above equation gives

$$\begin{aligned}
0 &= \left[\left(\frac{1}{4} + \beta\gamma_{2,t+1}\right)\pi_t + \frac{1}{4}\pi_{t-1} + \beta\gamma_{3,t+1}\rho u_t\right]\frac{\kappa}{1-\beta a_{t+1}} + \\
&\quad +\lambda\left(\frac{1-\beta a_{t+1}}{\kappa}\pi_t - \frac{1+\beta\rho b_{t+1}}{\kappa}u_t\right), \\
0 &= \left[\frac{\kappa\left(1+4\beta\gamma_{2,t+1}\right)}{4\left(1-\beta a_{t+1}\right)} + \frac{\lambda\left(1-\beta a_{t+1}\right)}{\kappa}\right]\pi_t + \frac{\kappa}{4\left(1-\beta a_{t+1}\right)}\pi_{t-1} \\
&\quad + \frac{\beta\gamma_{3,t+1}\rho\kappa^2 - \lambda\left(1+\beta\rho b_{t+1}\right)\left(1-\beta a_{t+1}\right)}{\kappa\left(1-\beta a_{t+1}\right)}u_t, \\
0 &= \left(\frac{\kappa^2\left(1+4\beta\gamma_{2,t+1}\right) + 4\lambda\left(1-\beta a_{t+1}\right)^2}{4\kappa\left(1-\beta a_{t+1}\right)}\right)\pi_t + \frac{\kappa}{4\left(1-\beta a_{t+1}\right)}\pi_{t-1} \\
&\quad + \frac{\beta\gamma_{3,t+1}\rho\kappa^2 - \lambda\left(1+\beta\rho b_{t+1}\right)\left(1-\beta a_{t+1}\right)}{\kappa\left(1-\beta a_{t+1}\right)}u_t.
\end{aligned}$$

Solving for π_t gives

$$\begin{aligned}
\pi_t &= -\frac{\kappa^2}{\kappa^2\left(1+4\beta\gamma_{2,t+1}\right) + 4\lambda\left(1-\beta a_{t+1}\right)^2}\pi_{t-1} \\
&\quad + \frac{4\lambda\left(1+\beta\rho b_{t+1}\right)\left(1-\beta a_{t+1}\right) - 4\beta\gamma_{3,t+1}\rho\kappa^2}{\kappa^2\left(1+4\beta\gamma_{2,t+1}\right) + 4\lambda\left(1-\beta a_{t+1}\right)^2}u_t. \qquad \text{(A.14)}
\end{aligned}$$

From the guessed value function we have

$$\frac{\partial V_{t-1}\left(\pi_{t-1}, u_t\right)}{\partial \pi_{t-1}} = \gamma_{2,t}\pi_{t-1} + \gamma_{3,t}u_t.$$

Also, the envelope theorem applied on the value function gives

$$\begin{aligned}
\frac{\partial V_{t-1}\left(\pi_t, u_{t+1}\right)}{\partial \pi_{t-1}} &= \frac{\pi_t + \pi_{t-1}}{4} \\
&= \frac{a_t\pi_{t-1} + b_t u_t + \pi_{t-1}}{4} \\
&= \frac{\left(1+a_t\right)}{4}\pi_{t-1} + \frac{b_t}{4}u_t.
\end{aligned}$$

Thus it must be that

$$\begin{aligned}\gamma_{2,t+1} &= \frac{1+a_{t+1}}{4}, \\ \gamma_{3,t+1} &= \frac{b_{t+1}}{4}.\end{aligned}$$

Substituting these expressions into (A.14) then gives

$$\pi_t = -\frac{\kappa^2}{\kappa^2\left(1+\beta\left(1+a_{t+1}\right)\right)+4\lambda\left(1-\beta a_{t+1}\right)^2}\pi_{t-1}+\frac{4\lambda\left(1+\beta\rho b_{t+1}\right)\left(1-\beta a_{t+1}\right)-\beta b_{t+1}\rho\kappa^2}{\kappa^2\left(1+\beta\left(1+a_{t+1}\right)\right)+4\lambda\left(1-\beta a_{t+1}\right)^2}u_t.$$

Since $\pi_t = a_t\pi_{t-1} + b_t u_t$, we arrive at

$$\begin{aligned}a_t &= -\frac{\kappa^2}{\kappa^2\left(1+\beta\left(1+a_{t+1}\right)\right)+4\lambda\left(1-\beta a_{t+1}\right)^2}, \\ b_t &= \frac{4\lambda\left(1-\beta a_{t+1}\right)\left(1+\beta\rho b_{t+1}\right)-\beta\rho\kappa^2 b_{t+1}}{\kappa^2\left(1+\beta\left(1+a_{t+1}\right)\right)+4\lambda\left(1-\beta a_{t+1}\right)^2},\end{aligned}$$

i.e. two equations that define a simultaneous recursion for a and b. The stationary solutions will then imply

$$\pi_t = a\pi_{t-1} + bu_t. \tag{A.15}$$

In order to compare this solution with the other cases, it is convenient to define $(\overline{a}-1) \equiv a$. Thus we will use

$$\pi_t = \left(\overline{a}-1\right)\pi_{t-1} + bu_t. \tag{A.16}$$

To get an expression for x_t, use (A.12):

$$\begin{aligned}x_t &= \frac{1-\left(\overline{a}-1\right)\beta}{\kappa}\pi_t - \frac{1+\beta\rho b}{\kappa}u_t \\ &= \left(\frac{1-\left(\overline{a}-1\right)\beta}{\kappa}\right)\left(\left(\overline{a}-1\right)\pi_{t-1}+bu_t\right) - \frac{1+\beta\rho b}{\kappa}u_t \\ &= \left(\frac{\left(\overline{a}-1\right)\left(1-\left(\overline{a}-1\right)\beta\right)}{\kappa}\right)\pi_{t-1} + \frac{b\left(1-\beta\left(\overline{a}-1\right)\right)-1-\beta\rho b}{\kappa}u_t \\ &= -\left(\frac{\left(\overline{a}-1\right)\left(\overline{a}\beta-\beta-1\right)}{\kappa}\right)\pi_{t-1} - \left(\frac{1-b\left(1+\beta\left(1-\overline{a}-\rho\right)\right)}{\kappa}\right)u_t,\end{aligned}$$

or

$$x_t = -c\pi_{t-1} - du_t.$$

Variances

To find the variance of inflation, note that

$$\mathrm{var}\left\{\pi_t\right\} = \left(\overline{a}-1\right)^2 \mathrm{var}\left\{\pi_{t-1}\right\} + b^2 \mathrm{var}\left\{u_t\right\} + 2\left(\overline{a}-1\right) b \ \mathrm{cov}\left\{\pi_{t-1}, u_t\right\}.$$

Since

$$\begin{aligned}
\mathrm{cov}\left\{\pi_{t-1}, u_t\right\} &= \mathrm{cov}\left\{\left(\overline{a}-1\right)\pi_{t-2} + bu_{t-1}, \rho u_{t-1} + \varepsilon_t\right\} \\
&= \left(\overline{a}-1\right)\rho \ \mathrm{cov}\left\{\pi_{t-2,}, u_{t-1}\right\} + b\rho \ \mathrm{var}\left\{u_t\right\} \\
&\Downarrow \\
\mathrm{cov}\left\{\pi_{t-1}, u_t\right\} &= \frac{b\rho}{1-\overline{a}\rho+\rho} \ \mathrm{var}\left\{u_t\right\},
\end{aligned}$$

due to stationarity. Thus

$$\begin{aligned}
\overline{a}\left(2-\overline{a}\right) \mathrm{var}\left\{\pi_t\right\} &= \left[b^2 + \frac{2\left(\overline{a}-1\right) b^2 \rho}{1-\left(\overline{a}-1\right)\rho}\right] \mathrm{var}\left\{u_t\right\}, \\
\mathrm{var}\left\{\pi_t\right\} &= \left[\frac{1+\overline{a}\rho-\rho}{\left(1-\overline{a}\rho+\rho\right)\overline{a}\left(2-\overline{a}\right)}\right] b^2 \mathrm{var}\left\{u_t\right\}.
\end{aligned}$$

Next, to find the variance of output, note that

$$\begin{aligned}
\mathrm{var}\left\{x_t\right\} &= c^2 \ \mathrm{var}\left\{\pi_{t-1}\right\} + d^2 \ \mathrm{var}\left\{u_t\right\} + 2cd \ \mathrm{cov}\left\{\pi_{t-1}, u_t\right\} \\
&= \left[\frac{1+\overline{a}\rho-\rho}{\left(1-\overline{a}\rho+\rho\right)\overline{a}\left(2-\overline{a}\right)} b^2 c^2 + d^2 + \frac{2bcd\rho}{\left(1-\overline{a}\rho+\rho\right)}\right] \mathrm{var}\left\{u_t\right\} \\
&= \left[\frac{b^2 c^2 \left(1+\overline{a}\rho-\rho\right) + \overline{a}\left(2-\overline{a}\right) d \left[d\left(1-\overline{a}\rho+\rho\right) + 2bc\rho\right]}{\left(1-\overline{a}\rho+\rho\right)\overline{a}\left(2-\overline{a}\right)}\right] \mathrm{var}\left\{u_t\right\}.
\end{aligned}$$

B State-space representation and solution

General form

Following Söderlind (1999), we wish to express the problem in the following general form. The economy evolves according to

$$\begin{bmatrix} z_{1,t+1} \\ E_t z_{2,t+1} \end{bmatrix} = A \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} + Bx_t + \begin{bmatrix} \varepsilon_{t+1} \\ \mathbf{0} \end{bmatrix},$$

where $z_{1,t}$ is a vector of n_1 predetermined (backward looking) variables, $z_{2,t}$ a vector of n_2 non-predetermined (forward looking variables), x_t the policy instrument, and ε_{t+1} the vector of innovations to $z_{1,t}$ with covariance Σ.

The policy maker has a loss function of the form

$$L_0 = E_0 \sum_{t=0}^{\infty} \beta^t \left(z_t' Q z_t + 2 z_t' U x_t + x_t' R x_t \right),$$

where the vector $z_t \equiv (z_{1,t}, z_{2,t})$. It is sometimes convenient to define a vector Y_t of goal variables, whereby the period loss function may alternatively be expressed as

$$L_t = Y_t' K Y_t.$$

The vector Y_t of goal variables is a linear combination of the z_t and x_t vectors:

$$Y_t = G_z z_t + G_x x_t.$$

The solution assuming discretionary decision-making may be expressed as

$$z_{1,t+1} = M\, z_{1,t} + \varepsilon_{t+1},$$

where M is an $n_1 \times n_1$ matrix,

$$z_{2,t} = C\, z_{1,t},$$

where C is an $n_2 \times n_1$ matrix, and the control variable will be

$$x_t = F\, z_{1,t}.$$

The Simple Model in Section 3

The Phillips curve may be written

$$p_t - p_{t-1} = \beta\left(p_{t+1|t} - p_t\right) + \kappa x_t + u_t,$$

which, solving for $p_{t+1|t}$ gives

$$p_{t+1|t} = \frac{1+\beta}{\beta}p_t - \frac{1}{\beta}p_{t-1} - \frac{\kappa}{\beta}x_t - \frac{1}{\beta}u_t.$$

Thus the state-space representation of the model may be written as

$$\begin{bmatrix} u_{t+1} \\ p_t \\ p_{t+1|t} \end{bmatrix} = \begin{bmatrix} \rho & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{\beta} & -\frac{1}{\beta} & \frac{1+\beta}{\beta} \end{bmatrix} \begin{bmatrix} u_t \\ p_{t-1} \\ p_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{\kappa}{\beta} \end{bmatrix} x_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \end{bmatrix}.$$

Hence

$$z_{1,t} \equiv \begin{bmatrix} u_t \\ p_{t-1} \end{bmatrix}, \qquad z_{2,t} \equiv p_t$$

i.e. the price level is the forward-looking variable and the exogenous shock and the lagged price level are the predetermined variables. If price-level targeting is the objective, the loss function is

$$\begin{aligned} L_t^{PT} &\equiv p_t^2 + \lambda^{PT} x_t^2 \\ &= \begin{bmatrix} u_t & p_{t-1} & p_t \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ p_{t-1} \\ p_t \end{bmatrix} + \lambda^{PT} x_t^2 \\ &= z_t' Q^{PT} z_t + x_t' \lambda^{PT} x_t. \end{aligned}$$

If the objective is to target the one-period inflation rate, the loss funtion may be written

$$\begin{aligned} L_t^{IT1} &\equiv \pi_t^2 + \lambda^{IT1} x_t^2 \\ &= \begin{bmatrix} u_t & p_{t-1} & p_t \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_t \\ p_{t-1} \\ p_t \end{bmatrix} + \lambda^{IT1} x_t^2 \\ &= z_t' Q^{IT1} z_t + x_t' \lambda^{IT1} x_t. \end{aligned}$$

In order to account for average inflation targeting (i.e $j > 1$), is it convenient to rewrite the state-space representation as follows. Insert a vector of lagged price levels into the state vector:

$$\tilde{z}_t \equiv \begin{bmatrix} u_t & p_{t-j} & p_{t-j+1} & \cdots & p_{t-1} & p_t \end{bmatrix}',$$

which means that $\tilde{z}_t$ will be a $(2+j)\times 1$ vector. The general state-space representation is then

$$\tilde{z}_{t+1} = A_j \tilde{z}_t + B_j x_t + \varepsilon_{j,t+1}, \tag{B.1}$$

where the matrices A_j, B_j and $\varepsilon_{j,t+1}$ will vary depending on the value of j (zeros and ones are added in the appropriate places). For example, if $j=4$, then

$$\begin{aligned}
\tilde{z}_{t+1} &= A_4 \tilde{z}_t + B_4 x_t + \varepsilon_{4,t+1}, \\
&= \begin{bmatrix} \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{1}{\beta} & 0 & 0 & 0 & -\frac{1}{\beta} & \frac{1+\beta}{\beta} \end{bmatrix} \begin{bmatrix} u_t \\ p_{t-4} \\ p_{t-3} \\ p_{t-2} \\ p_{t-1} \\ p_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{\kappa}{\beta} \end{bmatrix} x_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

The loss function is now easily expressed in terms of this 'extended' state vector:

$$L^{ITj}(\overline{\pi}_{j,t}, x_t) = \tilde{z}_t' Q_j \tilde{z}_t + x_t' \lambda x_t,$$

where

$$Q_j \equiv \begin{bmatrix} 0 & 0_{1\times(j+1)} \\ 0_{(j+1)\times 1} & \begin{matrix} \frac{1}{q_j} & 0 & \cdots & -\frac{1}{q_j} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ -\frac{1}{q_j} & 0 & \cdots & \frac{1}{q_j} \end{matrix} \end{bmatrix},$$

and where $q_j = j^2$. For example, if $j=4$, then

$$\begin{aligned}
L^{IT4}(\overline{\pi}_{4,t}, x_t) &= \left[\frac{1}{4}(p_t - p_{t-4})\right]^2 + \lambda x_t^2 \\
&= \begin{bmatrix} u_t & p_{t-4} & p_{t-3} & p_{t-2} & p_{t-1} & p_t \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{16} & 0 & 0 & 0 & -\frac{1}{16} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{16} & 0 & 0 & 0 & \frac{1}{16} \end{bmatrix} \begin{bmatrix} u_t \\ p_{t-4} \\ p_{t-3} \\ p_{t-2} \\ p_{t-1} \\ p_t \end{bmatrix} + \lambda x_t^2 \\
&= \tilde{z}_t' Q_4 \tilde{z}_t + x_t' \lambda x_t.
\end{aligned}$$

C The gains from commitment in forward-looking models

The influential paper by Barro and Gordon (1983) explained how an *inflation bias* could arise when the central bank had an incentive to spring surprises on the private sector. They used a Neo-Classical supply curve (i.e. with predetermined expectations) and showed that a discretionary solution to the central banks problem resulted in an outcome with the same level of output, but a higher level of inflation than in the commitment solution. In forward-looking models, another kind of inefficiency arises, which is independent of the central bank's output goal. To provide an illustration of this so-called *stabilization bias*, consider the model used in Section 3. Private sector behavior is summarized by the New-Keynesian Phillips Curve

$$\pi_t = \beta\pi_{t+1|t} + \kappa x_t + u_t, \tag{C.1}$$

where the inflation rate π_t is determined by today's value of the output gap x_t, expected future inflation $\pi_{t+1|t}$, and an exogenous shock u_t; β and κ are positive coefficients. Here we will assume that the exogenous shock is serially uncorrelated. As before, the output gap is considered as being the instrument of the central bank. The central bank's objective is to minimize an infinite sum of discounted losses,

$$\min_{\{x_\tau\}_{\tau=t}^{\infty}} \; E_t\,(1-\beta)\sum_{\tau=t}^{\infty}\beta^{\tau-t}L_\tau, \tag{C.2}$$

where the period loss function is

$$L\left(\pi_t, x_t\right) \equiv \frac{1}{2}\left[\left(\pi_t-\pi^*\right)^2+\lambda\, x_t^2\right].$$

Consider now the response of inflation and output to a unit shock to u_t. Figure C1 shows these impulse-responses for two cases, commitment (the dotted line) and discretion (the solid line). Consider first the case of discretion. Since the shock is temporary, discretionary decision-making will result in inflation and output being back on target from $t = 2$ and onwards. After all, at that point the shock is a bygone, and there is no reason to take it into consideration in a discretionary setting – this is the time-consistent solution. The only period in which output and inflation deviate from their steady-state values is the period in which the shock occurs. Output is depressed in order to minimize (with due consideration to output variability, i.e. the value of the preference parameter λ matters here) the deviation of inflation from its target.

However, this is not the most efficient response, i.e. it does not minimize the loss function. The efficient response is given by the commitment responses. As is shown in Figure C2, output is kept below the natural level for several periods following the shock, since this will produce inflation rates that are below target – this is the much-emphasized persistence induced by commitment.[29] And the anticipation of these negative inflation rates improves the monetary policy trade-off today. To see this, consider equation (C.1): when the shock u_t takes on a positive value, inflationary expectations $\pi_{t+1|t}$ fall, which in turn helps the policy-maker – output does not have to be depressed as much, and inflation today does not rise as much, compared to the discretion case.

[Figure C1 about here.]

The milder responses of output and inflation under commitment compared to discretion as demonstrated in Figure C1 also implies that the variances of inflation and output are lower under commitment. Figure C2 displays the two variance frontiers (the convex curves), and the variance frontiers obtained with a price level target and an average inflation target, plus a few negatively sloped straight lines that are to be explained below. The commitment frontier lies closest to the origin (and as mentioned in the main text, the price level target lies very close). The desireability of being closest to the origin can be seen in several equivalent ways. Easiest is perhaps to say that for any given level of inflation variance, output variance will be lower with commitment than with discretion, which given the form of the loss function results in lower overall loss.

[Figure C2 about here.]

An equivalent proof starts from a convenient way of rewriting the central bank's objective function. As $\beta \to 1$ the scaled loss function in (C.2) approaches the unconditional mean of the period loss

$$E\{L_t\} = var\{\pi\} + \lambda var\{x\}. \tag{C.3}$$

[29] To emphasize this point: even though the shock is temporary, the responses are persistent.

In $var\{\pi\} - var\{x\}$ space equation (C.3) describes what may be labelled an iso-loss curve. Thus it gives combinations of $var\{\pi\}$ and $var\{x\}$ that imply the same level of loss. The slope of the iso-loss is -λ^{-1}. Figure C2 provides three examples of iso-losses.

Consider the iso-loss curve closest to the origin.[30] This straight line represents the preferences of the central bank, while the convex (commitment) frontier may be interpreted as the opportunity set. Thus points outside the variance frontier are inefficient, while points inside it are infeasible. As usual, the optimal solution is to be found at the point of tangency. The overall loss at this optimal point can be read off the horizontal axis at the point where the iso-loss intersects it.

Consider next the iso-loss farthest from the origin, which is tangent to the discretion frontier. Following this iso-loss to the horizontal axis, it is easily seen that the discretion solution does indeed result in a higher overall loss (the iso-loss intersects the horizontal axis to the right of the first iso-loss curve). Finally, the intermediate iso-loss, tangent to the $j = 2$ variance frontier, shows that average inflation targeting provides an overall loss that is lower than $j = 1$.

[30] This line has been constructed assuming a particular value of λ.

Figure 1: Variance Frontiers in Simple New-Keynesian Model

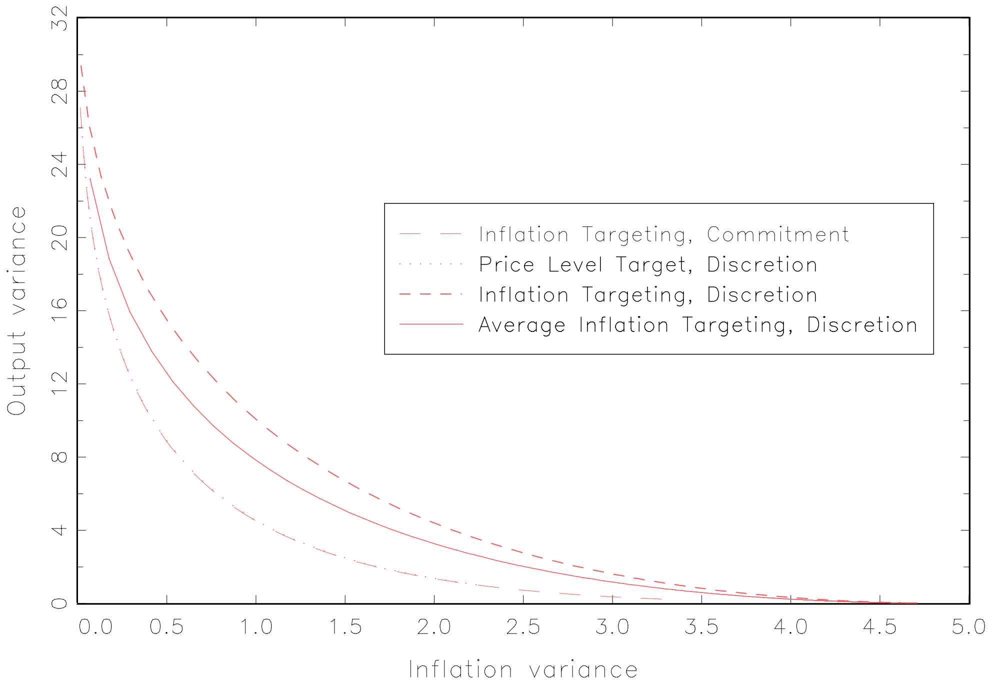

Figure 2: Dynamic response of inflation

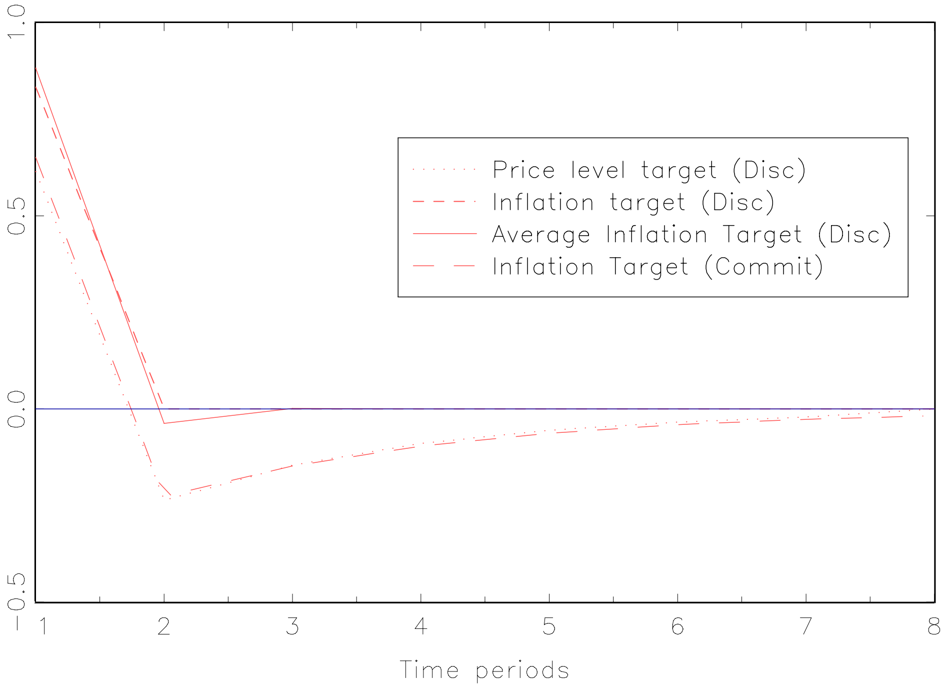

Figure 3: Dynamic response of output

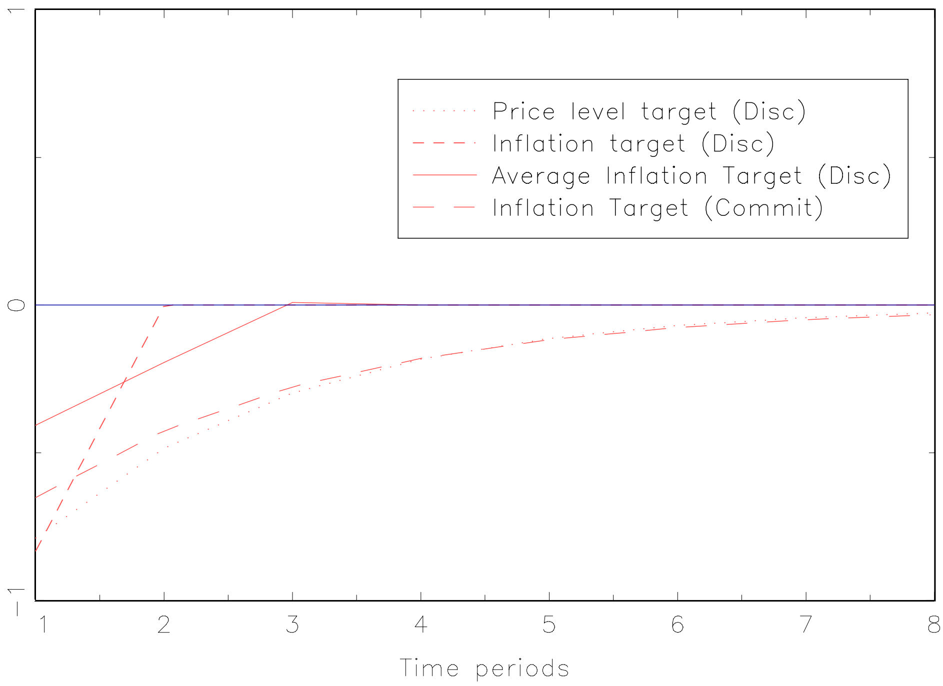

Figure 4: Discretion: Variance frontiers with different modified loss functions

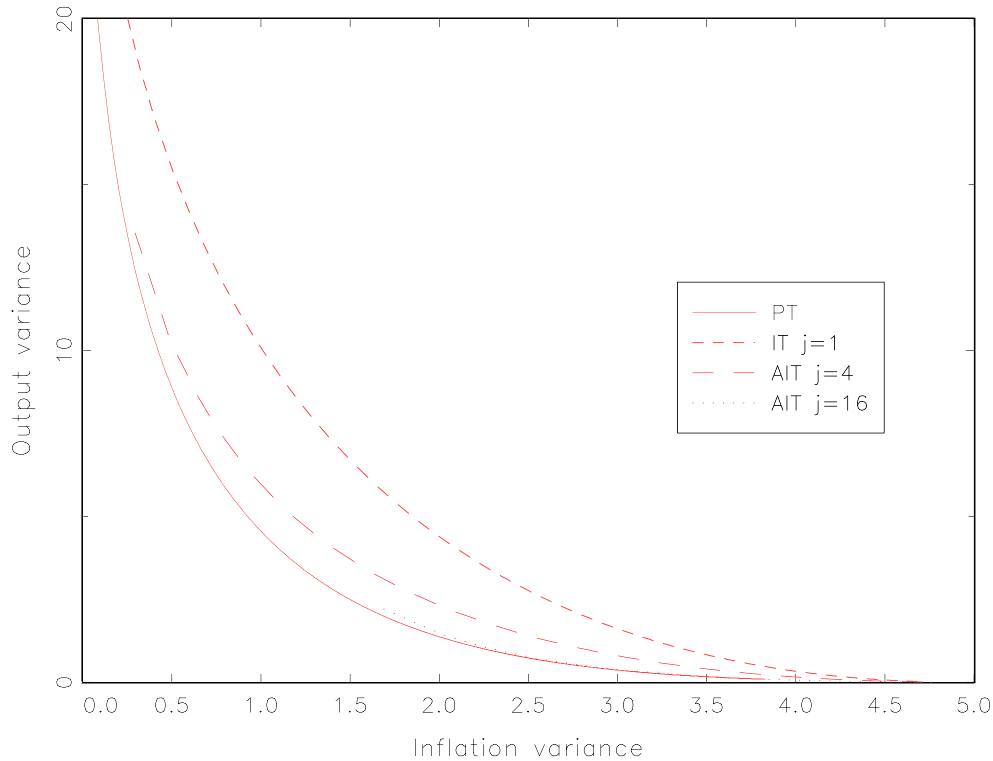

Figure 5: Dynamic response of inflation under discretion

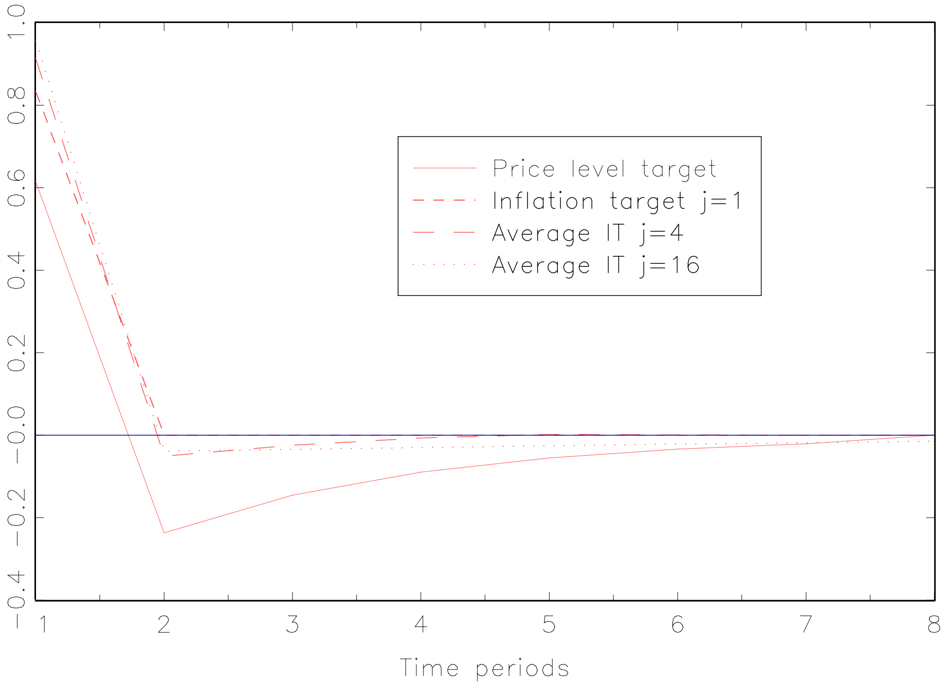

Figure 6: Dynamic response of output under discretion.

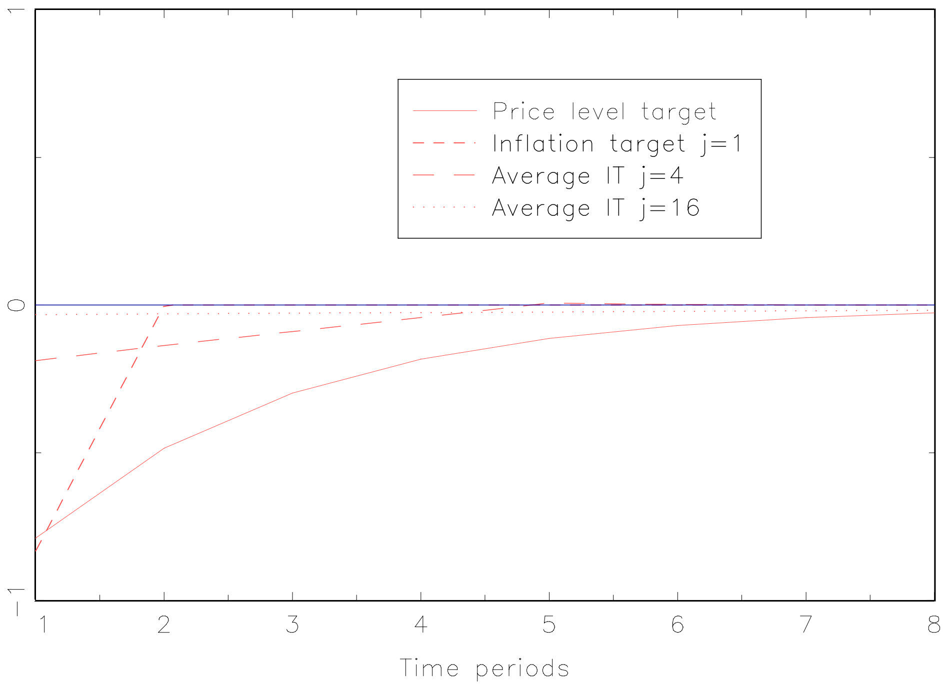

Figure 7: Hybrid model. Variance frontiers for different values of α.

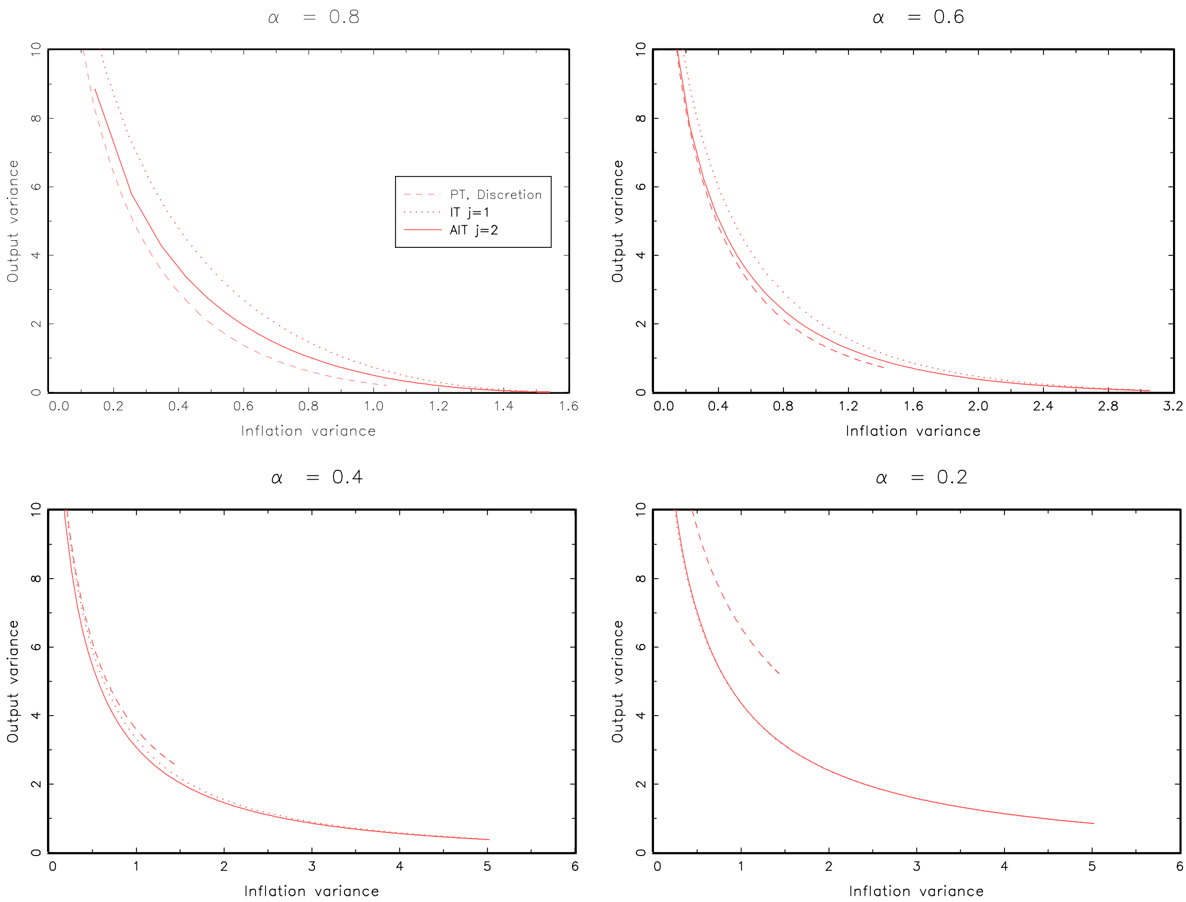

Figure C.1: Dynamic responses following shock. Commitment and discretion.

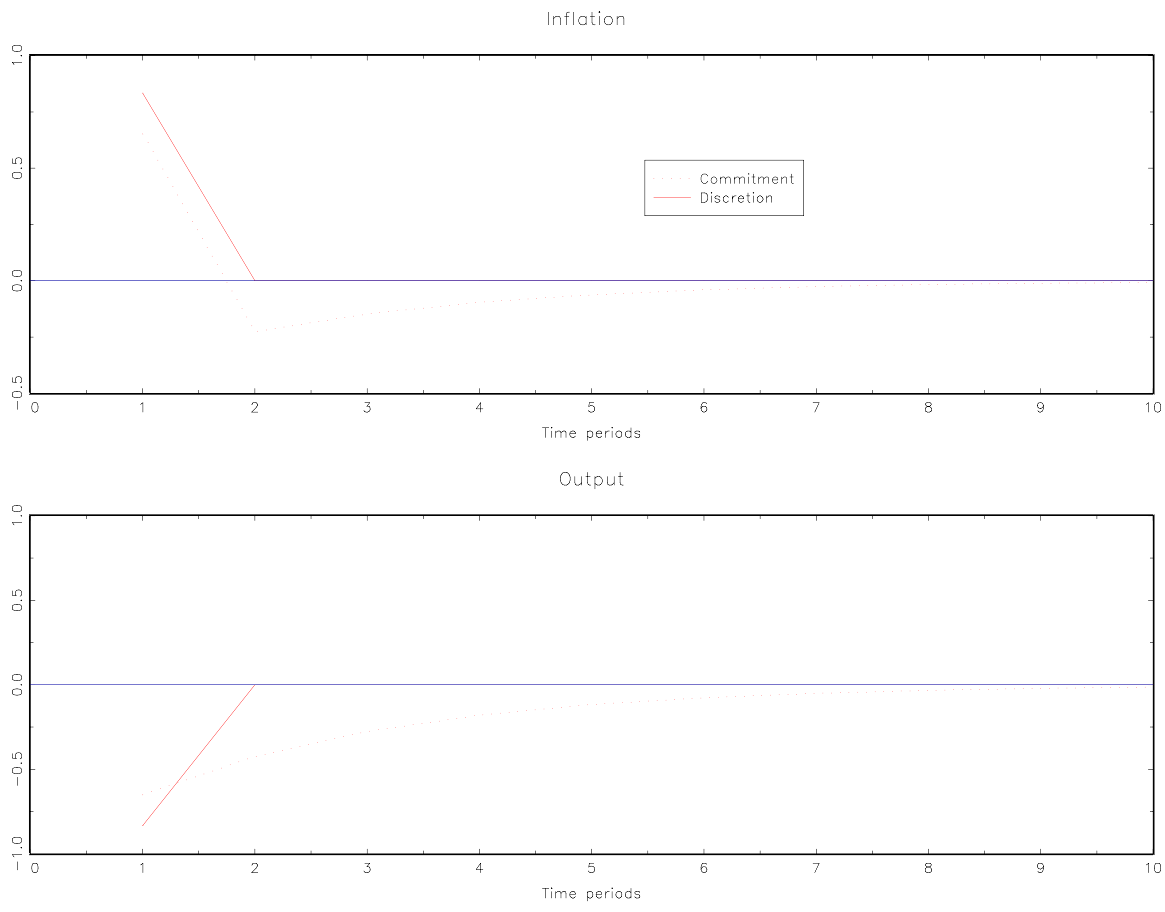

Figure C.2: Variance frontiers and iso-losses (when $\lambda = 0.2$)