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BAYESIAN INFERENCE IN STRUCTURAL SECOND-PRICE COMMON VALUE AUCTIONS

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ABSTRACT. Structural econometric auction models with explicit game-theoretic modeling of bidding strategies have been quite a challenge from a methodological perspective, especially within the common value framework. We develop a Bayesian analysis of the hierarchical Gaussian common value model with stochastic entry introduced by Bajari and Hortaçsu (2003). A key component of our approach is an accurate and easily interpretable analytical approximation of the equilibrium bid function, resulting in a fast and numerically stable evaluation of the likelihood function. The analysis is also extended to situations with positive valuations using a hierarchical Gamma model. We use a Bayesian variable selection algorithm that simultaneously samples the posterior distribution of the model parameters and does inference on the choice of covariates. The methodology is applied to simulated data and to a carefully collected dataset from eBay with bids and covariates from 1000 coin auctions. It is demonstrated that the Bayesian algorithm is very efficient and that the approximation error in the bid function has virtually no effect on the model inference. Both models fit the data well, but the Gaussian model outperforms the Gamma model in an out-of-sample forecasting evaluation of auction prices.

KEYWORDS: Bid function approximation, eBay, Internet auctions, Likelihood inference, Markov Chain Monte Carlo, Normal valuations, Variable selection.

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1. Introduction

Strategic bidding behavior in auctions has been a widely studied phenomenon since the pioneering work of Vickrey (1961), particularly over the last decades, see e.g. Wolfstetter (1996), Klemperer (1999, 2004), and Milgrom (2004) for recent surveys and a general introduction. The advances in auction theory have also found their way into the econometric analysis of auction data. It seems to be widely accepted that an explicit modeling of bidders' strategic considerations is a necessary condition for making economic sense of the observed patterns in the bids. The availability of high quality auction data has increased in recent years, especially with the advent of Internet auction sites, such as eBay, see e.g. Bajari and Hortaçsu (2004) for a survey. Paarsch (1992) and Elyakime, Laffont, Loisel and Vuong (1997) are two excellent examples of structural econometric analyses of auction data. Bajari (2005) and Paarsch and Hong (2006) survey the field.

Analyzing auction data through the lens of a structural game-theoretic model is not an easy task however. It has been quite a challenge to derive the strategic equilibrium bid function, i.e. the map from a bidder's conceived or estimated value of the object to their optimal bids, under realistic model assumptions. When such results are available they typically come in a form unamendable to analytical computations, and one needs to resort to time-consuming and possibly unstable numerical methods, such as numerical integration. This is an important obstacle to statistical inference as a single likelihood evaluation typically requires repeated evaluation of the inverse bid function for all bids in the dataset.

Common value auctions have been especially difficult to analyze by structural econometric models. In common value auctions, the auctioned object has the same value to every bidder, but the common value is unknown. The bidders use private information (their signal) to infer the unknown value. In an influential paper, Bajari and Hortaçsu (2003), henceforth BH, make a number of important advances that substantially simplify the analysis of data from common value auctions. BH prove that it is sufficient to compute the bid function in a selected auction and then extrapolate linearly to the other auctions in the dataset. This property speeds up the computation of the bid function considerably, and thereby also likelihood evaluations. BH also show that it is optimal to place a bid in the very last seconds of commonly used Internet auction formats, such as eBay's. This in turn implies that Internet auctions can be modeled as sealed bid auctions without additional strategic considerations of the timing of the bids. BH also extend the results in Milgrom and Weber (1982) to the situation with a stochastic number of bidders, an inherent feature of Internet auctions.

Our paper refines and extends the analysis in BH. Our first contribution is an accurate linear approximation of the equilibrium bid function, both for the case with a fixed and a stochastic number of bidders. The approximate bid function is of a particularly simple analytical form with interesting interpretation. It can be inverted and differentiated analytically, two extremely valuable properties for fast and numerically stable evaluations of the likelihood function.

An interesting aspect of the model in BH is the use of auction-specific covariates, both in the model for the common value and in the stochastic entry process. Our second contribution is the use of a highly efficient general posterior sampling algorithm that simultaneously approximates the joint posterior distribution of the model parameters and does Bayesian variable selection among the model's covariates. This allows us to quantify the importance of the individual covariates in the different parts of the model, and to correctly account for the uncertainty in the choice of covariates in e.g. the predictive distribution of the price. Bayesian variable selection also makes it possible to use a large number of covariates in the model since it typically reduces the dimensionality of the parameter space dramatically in every step of the Metropolis-Hasting algorithm, see Section 3.2.

The model in BH assumes a Gaussian distribution, both for the common value and for the bidders' signals around that common value. The Gaussian distribution is convenient, but has the obvious drawback of allowing negative values and signals. We also consider also an alternative model with non-negative values and signals following the Gamma distribution. An analytical approximation to the bid function is derived for this model too, paralleling the results for the Gaussian model. We propose a full Bayesian analysis of both the Gaussian and the Gamma models.

Finally, we apply the methodology to a newly collected dataset with bids and auction-specific information from 1000 eBay coin auctions. The data set was collected by careful human visual inspection of both pictures of the auctioned object and the seller's text description. Both the Gaussian and the Gamma models are shown to fit the data well. The Gaussian model also outperforms the Gamma model in an out-of-sample prediction evaluation on a sample with 48 auctions. The variable selection shows that the publicly available book value and the condition of the auctioned object are important determinants of bidders valuations, while eBay's detailed seller information, such as bidders' subjective ratings of sellers and the sellers' historical selling volumes, is essentially ignored by the bidders. The seller's posted minimum bid acts as a safeguard for the seller to avoid large losses. We show that it is typically optimal for the seller to post a minimum bid only slightly below the seller's valuation of the object if it remains unsold, despite the fact that a high minimum bid discourages auction entry. The estimation results are shown to be robust to a variety of modifications of the basic model.

2. Two models for second-price common value auctions

2.1. **General setup.** Assume the seller sets a publicly announced *minimum bid (reservation price)*, $r \geq 0$, and risk-neutral bidders compete for a single object using the same bidding strategy (symmetric equilibrium). The value of the object, v, is unknown and the same for each bidder at the time of bidding, but a prior distribution for v is shared by the bidders. To estimate v, each bidder relies upon their own private information of the object modeled as a private signal x from a distribution $x|v$ that is the same for all bidders. Let $f_v(v)$ denote the probability density function of v, $f_{x|v}(x|v)$ the conditional probability density function of $x|v$, and $F_{x|v}(x|v)$ the conditional cumulative distribution function of $x|v$. Since the auction involves symmetric bidders and a symmetric equilibrium we can focus on a single bidder without loss of generality. The bid function can be written (BH)

$$b(x,\lambda) = \begin{cases} \dfrac{\sum_{n=2}^{\infty}(n-1)\cdot p_{n-1}(\lambda)\cdot \int_{-\infty}^{\infty} v\cdot F_{x|v}^{n-2}(x|v)\cdot f_{x|v}^2(x|v)\cdot f_v(v)\ dv}{\sum_{n=2}^{\infty}(n-1)\cdot p_{n-1}(\lambda)\cdot \int_{-\infty}^{\infty} F_{x|v}^{n-2}(x|v)\cdot f_{x|v}^2(x|v)\cdot f_v(v)\ dv}, & \text{if } x \geq x^\star \\ 0 \text{ otherwise,} \end{cases} \tag{2.1}$$

where $p_{n-1}(\lambda)$ is the Poisson probability of $(n-1)$ bidders in the auction with λ as the expected value in the Poisson entry process. Bidders participate with a positive bid if their signal, x, is above the cut-off signal level $x^\star$. Given an arbitrary bidder with signal x, let y be the maximum signal of the other $(n-1)$ bidders. The cut-off signal level is then given in implicit form as (Milgrom and Weber, 1982)

$$x^\star(r,\lambda) = \inf_x \left(E_n E[v|X=x, Y<x, n] \geq r\right),$$

which gives the minimum bid, r, as

$$r(x^\star,\lambda) = \sum_{n=1}^{\infty} p_n(\lambda)\cdot \frac{\int_{-\infty}^{\infty} v\cdot F_{x|v}^{n-1}(x^\star|v)\cdot f_{x|v}(x^\star|v)\cdot f_v(v)\ dv}{\int_{-\infty}^{\infty} F_{x|v}^{n-1}(x^\star|v)\cdot f_{x|v}(x^\star|v)\cdot f_v(v)\ dv}. \tag{2.2}$$

The minimum bid is exogenously given by the seller and $x^\star$ is then given as the solution to (2.2).

In both the Gaussian and the Gamma models presented below the expected value μ and the variance σ^2 exists in the distribution of v. Similar to BH we specify regression models for $(\mu_j, \sigma_j^2, \lambda_j)$ in auction j as

$$\begin{aligned} \mu_j &= z'_{\mu j}\beta_\mu \\ \sigma_j^2 &= \exp\left(z'_{\sigma j}\beta_\sigma\right) \\ \lambda_j &= \exp\left(z'_{\lambda j}\beta_\lambda\right), \end{aligned} \tag{2.3}$$

where $z_j = (z'_{\mu j}, z'_{\sigma j}, z'_{\lambda j})'$ are auction specific covariates in auction j. Also, let $\beta = (\beta_\mu, \beta_\sigma, \beta_\lambda)'$. BH models σ^2 with a linear function of covariates and therefore need to restrict the elements of β_σ to ensure that σ^2 is positive, while β_σ is unrestricted in our setup.

BH makes the assumption that bids in parallel auctions are independent, and show that last-minute bidding is a symmetric Nash equilibrium on eBay. This allows us to model eBay auctions as independent second-price auctions. The likelihood function of bids is complicated since some bids are unobserved. First, some bidders may draw a signal $x < x^\star$, in which case they do not place a bid. Second, the highest bid is usually not observed because of eBay's *proxy bidding system*, see BH and Section 4.1 for more details. The bid distribution for a single auction is of the form:

$$f_b\left(b|\beta,\eta,r,z,v\right) = f_{x|v}\left[\phi\left(b\right)|\beta,\eta,r,z,v\right]\phi'\left(b\right), \tag{2.4}$$

where $\phi(b)$ is the inverse bid function, and η is a vector of additional parameters in the model. Let n be the number of bidders who submit a positive bid in a given auction and let $\mathbf{b} = (b_2, b_3, \ldots, b_n)$ be the vector of observed bids, where $b_2 > b_3 > \ldots > b_n$. Then, the likelihood function for that auction is given by

$$\begin{aligned} &f_{\mathbf{b}}\left(b_2, b_3, \ldots, b_n|\beta,\eta,r,z\right) \\ &= \sum_{i=n}^{\bar{N}} p_i(\lambda) \cdot \int_{-\infty}^{\infty} F_{x|v}\left(x^\star|\beta,\eta,v\right)^{i-n} \cdot \left\{1 - F_{x|v}\left[\phi\left(b_2\right)|\beta,\eta,v\right]\right\}^{I(n\geq 1)} \\ &\qquad \times \prod_{i=2}^{n} f_b\left(b_i|\beta,\eta,r,z,v\right) \cdot f_v(v|\beta)dv, \end{aligned} \tag{2.5}$$

where $I(n \geq 1)$ is an indicator variable for at least one observed bid in the auction, and $\bar{N}$ is an upper bound for the total number of potential bidders. Following BH, $\bar{N}$ is set to 30. If $n = 1$, b_2 equals the minimum bid r.

We use Bayesian methods to estimate the model, see Section 3. A single evaluation of the posterior density (likelihood function) requires numerical integration to compute $b\left(x|\beta,r,z\right)$ in (2.1), followed by additional numerical work to invert and differentiate $b\left(x|\beta,r,z\right)$. The same applies to the computation of $x^\star$. This costly procedure needs to be repeated for each of the auctions in the dataset. BH cleverly exploit a linearity property of the bid function that confines a large portion of the numerical work to a single auction which is then extrapolated linearly to the other auctions. Nevertheless, the likelihood evaluation suggested by BH is not fast enough to be routinely used for inference. Instead, we make use of analytical approximations of the bid functions, see Section 2.2 and 2.3, which lead to much faster and numerically more stable likelihood evaluations.

Paarsch (1992) use an interesting method initially suggested by Levin and Smith (1991) to compute bid functions for a class of models including a model with Gaussian valuations (see the next subsection). This method assumes a diffuse prior for the auction value, which makes it harder to use an interesting covariate structure as in (2.3). Moreover, contrary to our approximation approach, the Levin-Paarsch-Smith (1991) method is not easily extended to the case with stochastic auction entry.

2.2. **Gaussian model.** Let v_j denote the common value in auction j, and let x_{ij} denote the signal of the ith bidder in auction j. The hierarchical Gaussian model in BH can be defined as

$$\begin{aligned} v_j &\sim N(\mu_j, \sigma_j^2), \;\; j = 1, ..., m, \\ x_{ij}\,|v_j &\overset{iid}{\sim} N(v_j, \kappa\sigma_j^2), \;\; i = 1, ..., n_j, \end{aligned}$$

where m is the total number of auctions, and n_j is the number of bidders that bid zero or place a positive bid in auction j. We derive the following linear approximation of the bid function (see Appendix A for details)

$$b(x, \lambda) \approx \begin{cases} c + \omega\mu + (1-\omega)x, \text{ if } x \geq x^\star \\ \\ 0 \text{ otherwise} \end{cases} \tag{2.6}$$

where $c = -\frac{\sqrt{\kappa}\sigma\gamma\theta(\lambda-2)}{\gamma(\lambda-2)+1+\frac{\kappa}{2}}$, $\omega = \frac{\frac{\kappa}{2}}{\gamma(\lambda-2)+1+\frac{\kappa}{2}}$, $\theta = 1.96$ and $\gamma = 0.1938$. The cutoff-signal can be similarly approximated by

$$x^\star(r, \lambda) \approx \frac{r - \sum_{n=1}^{\infty} p_n(\lambda)(\tilde{c} + \tilde{\omega}\mu)}{\sum_{n=1}^{\infty} p_n(\lambda)(1-\tilde{\omega})}, \tag{2.7}$$

where $\tilde{c} = -\frac{\sqrt{\kappa}\sigma\gamma\theta(n-1)}{\gamma(n-1)+\frac{1}{2}+\frac{\kappa}{2}}$, and $\tilde{\omega} = \frac{\frac{\kappa}{2}}{\gamma(n-1)+\frac{1}{2}+\frac{\kappa}{2}}$.

The approximate bid function has the following properties. First, it is a linear combination of the signal x and the publicly held value μ. The weight ω increases monotonically towards 1 as κ increases, which gives

$$b(x) \longrightarrow x \; if \; \kappa \longrightarrow 0, \;\; \text{and} \;\; b(x) \longrightarrow \mu \; if \; \kappa \longrightarrow \infty.$$

The higher precision in signals the more the bidders trust their private information and vice versa. A higher variance of the common value v implies a higher risk of drawing a large signal and thus a higher risk of overestimating the true value of the object, why bidders should lower their bids. The approximate bid function captures this effect well, increasing the value of σ leads to lower bids. If $n = 2$, then $\Phi^{n-2}(t) = 1$, and the bid function in 2.1 can be computed exactly, and equals the approximate solution in 2.6.

The approximation in (2.6) can also be used to derive the unconditional distribution of the bids $\mathbf{b} = (b_1, ..., b_n)'$ in an auction with a given number of bids n. It is straightforward to show that

$$\mathbf{b} \sim N\left[(c+\mu)\mathbf{1}_n, (1-\omega)^2\sigma^2\left(\kappa\mathbf{I}_n + \mathbf{1}_{n\times n}\right)\right], \tag{2.8}$$

where $\mathbf{I}_n$ is the identity matrix, and $\mathbf{1}_n$ and $\mathbf{1}_{n\times n}$ denote the $n \times 1$ vector and the $n \times n$ matrix with ones, respectively. Note that we are here ignoring the truncation that comes from $x^\star$. At first glance one might think that κ is merely a factor that inflates the variance of the bids and is mainly estimated from the variance of the observed bids. Equation (2.8) shows that this is a too simplistic view. First, the unconditional variance of the bids, $(1-\omega)^2(\kappa+1)\sigma^2$, obviously increases with κ via the factor $(\kappa + 1)$, but κ also affects $\text{var}(b)$ through ω in a non-linear fashion that depends on λ. Second, κ determines the dependence between bids as the correlation between any pair of bids is $(1 + \kappa)^{-1}$.

Figure 2.1 compares the exact and approximate bid function graphically. The exact bid function is computed by numerical integration as in BH. The upper left sub-graph displays the

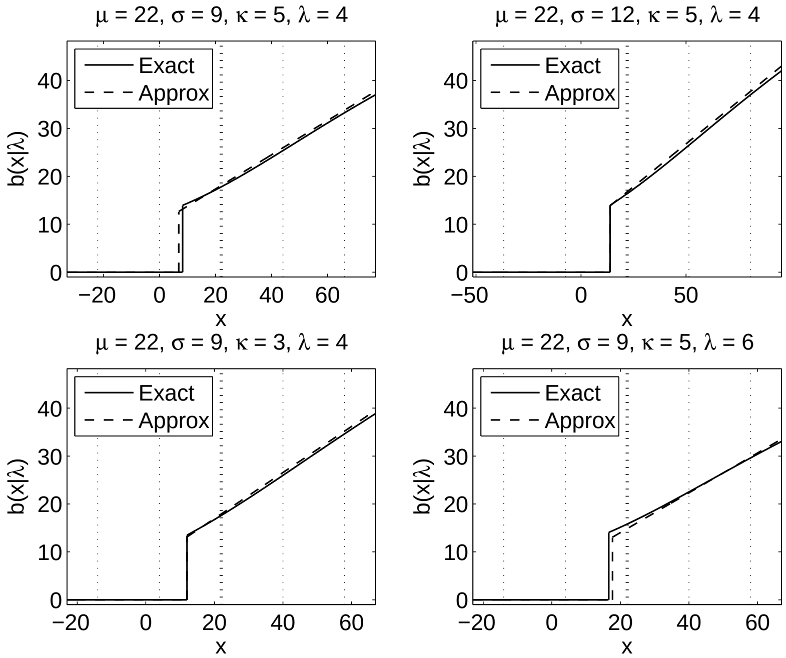

FIGURE 2.1. Examining the accuracy of the approximate bid function in the Gaussian model for different configurations of parameter values. The vertical lines in the figures mark out the mean (thick dotted) $\pm$ 1 and 2 standard deviations (thin dotted) in the unconditional distribution of the signals, x.

bid function and its approximation for a representative auction in the eBay data set analyzed in Section 4. The representative auction is based on the median of the covariates in the eBay data, analyzed in Section 4, and the posterior mean of the model parameters. Rounded to the nearest integer, this gives $\kappa = 5, \mu = 22, \sigma = 9, \lambda = 4$, and $\frac{r}{\mu} = 0.5$. The other sub graphs are variations from the representative auction. The approximation of the bid function is very good in all four cases. It is not easy to assess the importance of the approximation errors in Figure 2.1 for practical work, but experiments in Section 3.2 and 4.2 show that inferences based on the approximate bid function are very similar to those obtained from the exact bid function.

2.3. **Gamma model.** A drawback of the Gaussian model in the previous section is that the common value, the signals and the bids can be negative. It can be argued that the Gaussian distribution serves as a good approximation when the mean is at least a couple of standard deviations away from zero, which is a common situation in practice. This is clearly not always the case, however, and there are situations when it would be better to use a distribution with non-negative support. We will therefore extend and analyze a model in Gordy (1998) based on the Gamma distribution. The Gaussian and Gamma models are compared in the empirical application in Section 4.

The Gamma model is more conveniently written in terms of *inverse* signals, $s_{ij} = \frac{1}{x_{ij}}$,

$$v_j \sim Gamma(\xi_j, \psi_j), \;\; j = 1, ..., m, \;\; E(v_j) = \mu_j = \frac{\xi_j}{\psi_j}, \;\; Var(v_j) = \sigma_j^2 = \frac{\xi_j}{\psi_j^2}$$

$$s_{ij} \,|v_j \overset{iid}{\sim} Gamma(\tau, \tau v_j), \;\; i = 1, ..., n_j, \;\; E(s_{ij} \,|v_j) = \frac{1}{v_j}, \;\; Var(s_{ij} \,|v_j) = \frac{1}{\tau v_j^2}, \tag{2.9}$$

where τ is a precision parameter, m is the total number of auctions, and n_j is the number of bidders that bid zero or place a positive bid in auction j. The bid function for the Gamma model is of the form (see Gordy (1998) for the case with a fixed number of bidders)

$$(2.10) \qquad b(x,\lambda) = \begin{cases} \frac{\sum_{n=2}^{\infty}(n-1)\cdot p_{n-1}(\lambda)\cdot\int_0^{\infty} v\cdot(1-F_{s|v}(1/x|v))^{n-2}\cdot f_{s|v}^2(1/x|v)\cdot f_v(v)\ dv}{\sum_{n=2}^{\infty}(n-1)\cdot p_{n-1}(\lambda)\cdot\int_0^{\infty}(1-F_{s|v}(1/x|v))^{n-2}\cdot f_{s|v}^2(1/x|v)\cdot f_v(v)\ dv}, & \text{if } x \geq x^\star \\ 0, \text{ if } x < x^\star, \end{cases}$$

and the minimum bid function for the Gamma model is given by

$$(2.11) \qquad r(x^\star,\lambda) = \sum_{n=1}^{\infty} p_n(\lambda)\cdot \frac{\int_0^{\infty} v\cdot(1-F_{s|v}(1/x^\star|v))^{n-1}\cdot f_{s|v}(1/x^\star|v)\cdot f_v(v)\ dv}{\int_0^{\infty}(1-F_{s|v}(1/x^\star|v))^{n-1}\cdot f_{s|v}(1/x^\star|v)\cdot f_v(v)\ dv}.$$

Gordy (1998) obtains a finite series expansion of the bid function for the Gamma model (see $B_2(x)$ in equation 7 in Gordy's article). Gordy's solution for the bid function is elegant and fast, but there are two reasons why it has limited use in a likelihood-based approach. First, the likelihood function depends on the *inverse* bid function which has to be solved numerically for each bid in every auction. Second, the solution is restricted to the set of positive integers for τ, which makes it hard to link τ to covariates. The equilibrium bid function for the Gamma model can be approximated in a similar way to the Gaussian model, see Appendix B. The approximate bid function is given by

$$(2.12) \qquad b(x) \approx \frac{[\xi + 2\tau + (\lambda-2)\xi_\tau]\cdot x}{\psi x + 2\tau + (\lambda-2)\psi_\tau},$$

where

$$\xi_\tau = -0.1659 - 0.0159\cdot\tau + 0.2495\cdot\log(1+\tau),$$

and

$$\psi_\tau = 0.3095 - 0.0708\cdot\tau + 1.0241\cdot\log(1+\tau),$$

for $0.1 \leq \tau \leq 10$. If necessary, other values of τ can be handled similarly, see Appendix B for details. The bid function in (2.10) can be computed exactly when $\lambda = 2$ or $\tau = 1$ (Gordy, 1998), and is then identical to the approximate bid function in (2.12).

The approximate minimum bid function is given by

$$(2.13) \qquad r(x^\star,\lambda) \approx \frac{[\xi + \tau + (\lambda-1)\xi_\tau]\cdot x^\star}{\psi x^\star + \tau + (\lambda-1)\psi_\tau}.$$

The approximate bid function in Equation (2.12) has the following properties. First, the more precise is the public information (larger ψ) the less weight is placed on the bidder's private signal. To see this, replace ξ with $\psi\mu$ in equation (2.12) to obtain

$$(2.14) \qquad b(x) \approx \frac{\psi\mu + 2\tau + (\lambda-2)\xi_\tau}{\psi + \frac{2\tau+(\lambda-2)\psi_\tau}{x}},$$

which yields

$$b(x) \longrightarrow \mu \ if\ \psi \longrightarrow \infty \ \text{ and } \ b(x) \longrightarrow \frac{2\tau + (\lambda-2)\xi_\tau}{2\tau+(\lambda-2)\psi_\tau}\cdot x \ if\ \psi \longrightarrow 0.$$

The approximate bid function also satisfies

$$b(x) \longrightarrow \mu \ if\ \tau \longrightarrow 0,$$

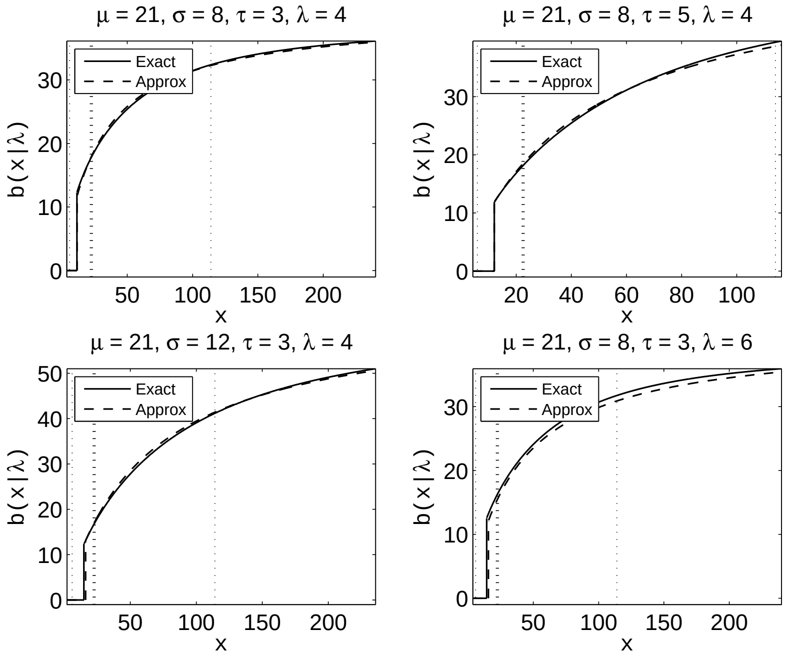

FIGURE 2.2. Examining the accuracy of the approximate bid function in the Gamma model for different configurations of parameter values. The vertical lines in the figures are mean (thick dotted), and the 5% and 95% percentiles (thin dotted) in the unconditional distribution of the signals, x.

and

$$b(x) \approx \frac{\left[2 + \frac{\xi}{\tau} + (\lambda - 2)\frac{\xi_\tau}{\tau}\right] \cdot x}{2 + \frac{\psi x}{\tau} + (\lambda - 2)\frac{\psi_\tau}{\tau}} \longrightarrow x \; if \; \tau \longrightarrow \infty.$$

The last two results are easily proved by noting that our approximation routine in Appendix B implies that

$$\left(\frac{\xi_\tau}{\tau}, \frac{\psi_\tau}{\tau}\right) \longrightarrow (0,0) \; if \; \tau \longrightarrow \infty,$$

and

$$(\xi_\tau, \psi_\tau) \longrightarrow (0,0) \; if \; \tau \longrightarrow 0.$$

Figure 2.2 compares the approximate bid function in Equation (2.12) to the exact bid function in Equation (2.10), using the parameter values from the representative eBay auction, which is defined analogously to the Gaussian model in Section 2.2. The representative auction has parameter values $\tau = 3, \mu = 21, \sigma = 8$, and $\lambda = 4$. The approximation deteriorates somewhat with increasing λ, but is still very accurate, irrespective of τ and σ.

2.4. **Discussion.** The two previous subsections have shown that our approximation method can be used to obtain accurate, easily computable and interpretable approximate bid functions for second-price common value auctions with Gaussian or Gamma distributed signals and values. It is obvious however that our approach cannot be used for any arbitrary auction setup and valuation structure. To state the conditions under which our approach can be used, let us first define a family of distributions $\mathcal{P}$ to be *closed under multiplication* if for any $p_1, p_2 \in \mathcal{P}$ we have that $k \cdot p_1 \cdot p_2 \in \mathcal{P}$, where k is a constant. The success of our approach hinges upon i) that the distribution function $F_{x|v}(x|v)$ can be well approximated by the kernel of a density function $p_{x|v}(x|v)$, ii) that $p_{x|v}(x|v)$, $f_{x|v}(x|v)$ and $f_v(v)$ all belong to a family of distributions $\mathcal{P}$ that is closed under multiplication (with respect to v), and iii) that the kernel of any member of $\mathcal{P}$ can be integrated analytically. Since the distribution function $F_{x|v}(x|v)$ appears in other major auction formats, it is clear that this approach can at least in principle also be used to approximate the bid function in other auction setups.

3. Bayesian inference

3.1. **Prior.** Bayesian inference combines the likelihood function in (2.5) with a prior distribution on the unknown model parameters. The numerical algorithms that we use for sampling from the joint posterior distribution (see next section) can be used with any prior. This section proposes a particular prior that can be used with very limited input from the user. Our prior for β_μ and β_σ in the Gaussian model is motivated by the fact that the common values $v = (v_1, ..., v_n)'$ are modeled as a heteroscedastic regression

$$v = Z_\mu \beta_\mu + \varepsilon,\ \ \varepsilon_i \sim N(0, \sigma_i^2), \tag{3.1}$$

where $\sigma^2 = (\sigma_1^2, ..., \sigma_n^2)' = \exp(Z_\sigma \beta_\sigma)$. Pre-multiplying both sides of (3.1) by the $n \times n$ diagonal matrix $D^{1/2} = \text{Diag}[\exp(-z'_{\sigma_1}\beta_\sigma/2), ..., \exp(-z'_{\sigma_n}\beta_\sigma/2)]$ we obtain the transformed homoscedastic regression

$$\tilde{v} = \tilde{Z}_\mu \beta_\mu + \tilde{\varepsilon},\ \ \tilde{\varepsilon}_i \overset{iid}{\sim} N(0, 1), \tag{3.2}$$

where $\tilde{v} = D^{1/2}v$, $\tilde{Z}_\mu = D^{1/2}Z_\mu$ and $\tilde{\varepsilon} = D^{1/2}\varepsilon$. We can now specify a g-prior (Zellner, 1986) for β_μ, conditional on β_σ, as

$$\beta_\mu|\beta_\sigma \sim N[0, c_\mu(\tilde{Z}'_\mu \tilde{Z}_\mu)^{-1}] = N[0, c_\mu(Z'_\mu D Z_\mu)^{-1}],$$

where $c_\mu > 0$ is a scaling factor that determines the tightness of the prior. Setting $c_\mu = n$, where n is the number of auctions in the sample, makes the information in the prior equivalent to the information in a single auction (conditional on β_σ), which is a useful benchmark. The marginal prior for β_σ is also taken to be a g-prior

$$\beta_\sigma \sim N[0, c_\sigma(Z'_\sigma Z_\sigma)^{-1}].$$

The Gamma model in Section 2.3 cannot be written as a heteroscedastic linear regression, and a similar characterization of the prior for β_μ and β_σ is therefore not possible. We will instead assume that $\beta_\mu \sim N[0, c_\mu(Z'_\mu Z_\mu)^{-1}]$ independently from $\beta_\sigma \sim N[0, c_\sigma(Z'_\sigma Z_\sigma)^{-1}]$ in the Gamma model.

Turning to the Poisson entry model, we use the following g-prior for β_λ

$$\beta_\lambda \sim N[0, c_\lambda(Z'_\lambda Z_\lambda)^{-1}].$$

We use an inverse Gamma prior for κ in the Gaussian model, $\kappa \sim IG(\bar{\kappa}, g)$, where $\bar{\kappa}$ is the prior mean of κ and g are the degrees of freedom. Similarly, we assume the prior $\tau \sim IG(\bar{\tau}, h)$ in the Gamma model.

We also allow for variable selection among the covariates, or equivalently that some regression coefficients are exactly zero. This is implemented by introducing point masses at zero in the prior distribution on the regression coefficients. For example, the prior on β_σ now follows the two-component mixture distribution

$$p(\beta_\sigma) = \begin{cases} N[0, c_\sigma(Z'_\sigma Z_\sigma)^{-1}] \text{ with probability } \pi_\sigma \\ \\ \qquad\qquad 0 \text{ otherwise,} \end{cases}$$

where π_σ is referred to as the prior inclusion probability. The user thus needs to specify the eight hyperparameters c_μ, c_σ, c_λ, $\pi_\mu, \pi_\sigma, \pi_\lambda$, $\bar{\kappa}$, and g. We typically set $c_\mu = c_\sigma = c_\lambda = c$ and $\pi_\mu = \pi_\sigma = \pi_\lambda = \pi$, thus reducing the number of prior hyperparameters to four. We document in Sections 3.2 and 4 that the posterior distribution and the variable selection inference are not overly sensitive to the exact choice of these prior hyperparameters, and that a good default value for c is $c = n$.

3.2. **A Metropolis-Hastings algorithm for variable selection.** It is clear from (2.5) that the likelihood function for second-price common value auctions is highly non-standard, so the posterior distribution of the model parameters cannot be analyzed by analytical methods. The most commonly used algorithm for simulating from posterior distributions is the Metropolis-Hastings (MH) algorithm, which belongs to the Markov Chain Monte Carlo (MCMC) family of algorithms, see e.g. Gelman et al. (2004) for an introduction. At a given step of the algorithm, a proposal draw β_p is simulated from the proposal density $f(\beta_p|\beta_c)$, where β_c is the current draw of the parameters (i.e. the most recently accepted draw). The proposal draw β_p is then accepted into the posterior sample with probability

$$a(\beta_c \to \beta_p) = \min\left[1, \frac{p(\beta_p|y)/f(\beta_p|\beta_c)}{p(\beta_c|y)/f(\beta_c|\beta_p)}\right],$$

where $p(\beta|y)$ denotes the posterior density. If β_p is rejected, then β_c is included in the posterior sample. This sampling scheme produces (autocorrelated) draws that converge in distribution to $p(\beta|y)$. The $f(\beta_p|\beta_c)$ can in principle be any density, but should for efficiency reasons be a fairly good approximation to the posterior density. One possibility is the *random walk Metropolis* algorithm where $f(\beta_p|\beta_c)$ is multivariate normal density with mean β_c and covariance matrix $-c{\cdot}H^{-1}$, where H is the Hessian matrix evaluated at the posterior mode and c is a scaling constant. This algorithm is used in BH. The random walk Metropolis is a robust algorithm, but is well known to be rather inefficient. Moreover, it is not easily extended to the case with variable selection. A more efficient alternative that also can be extended to variable selection is the *independence sampler* where $f(\beta_p|\beta_c)$ is the multivariate-t density $t(\hat{\beta}, -H^{-1}, h)$, where $\hat{\beta}$ is the posterior mode of $p(\beta|y)$, H is again the Hessian matrix at the mode and h is the degrees of freedom. The multivariate-t density is here defined in terms of its mean and covariance matrix. We typically use $h = 10$ degrees of freedom, which we have found to work well. The posterior mode and Hessian matrix can be easily obtained using a standard Newton-Raphson algorithm with BFGS update of the Hessian matrix (Fletcher, 1987).

Consider now setting a subset of the elements in $\beta = (\beta'_\mu, \beta'_\sigma, \beta'_\lambda)'$ to zero (any other value is also possible). In a regression situation, this is clearly equivalent to selecting a subset of the covariates. Let $\mathcal{J} = (j_1, ..., j_r)$ be a vector of binary indicators such that $j_i = 0$ iff the ith element of β is zero. We can view these indicators as a set of new parameters. We shall here for simplicity assume that the elements of $\mathcal{J}$ are independent a priori with $\Pr(j_i) = \pi$ for all i, so that π is the prior probability of including the ith covariate in the model, but other priors for $\mathcal{J}$ are just as easily handled. Appendix B describes in detail how this algorithm can be generalized to sample from the joint posterior distribution of the parameters and the variable selection indicators $\mathcal{J}$, all in a single MCMC run.

We use the mean acceptance probability and the inefficiency factor (IF) to measure the performance of the Metropolis-Hastings algorithm. The inefficiency factor is defined as $1 + 2\sum_{k=1}^{K} \rho_k$, where ρ_k is the autocorrelation at the kth lag in the MCMC chain for a given parameter and K is an upper limit of the lag length such that $\rho_k \approx 0$ for all $k > K$. The inefficiency factor approximates the ratio of the numerical variance of the posterior mean from the MCMC chain to that from hypothetical iid draws. Put differently, the IF measures the number of draws needed to obtain the equivalent result of a single independent draw. IFs close to unity is therefore an indication of a very efficient algorithm.

We conducted a simulation study to evaluate the performance of the variable selection procedure, where we are particularly interested in exploring the sensitivity of the posterior inclusion probabilities to changes in prior hyperparameters $c = c_\mu = c_\sigma = c_\lambda$, and g. We set the prior inclusion probability to $\pi = 0.2$ and $\bar{\kappa} = 0.25$ throughout the simulations. Rather than setting the parameters in the data generating model to arbitrary values we will here generate data from a model that mimics the estimated model for eBay data in BH. We simulated 50 full datasets, each with 407 auctions, using the posterior mean estimates in BH as parameter values. The covariates in the model were simulated independently to mimic the summary statistics in Table 1 and 2 in BH. The eBay auction model in BH for auction j, without a secret reserve price, can be written as (see BH for a description of the covariates)

$$\mu_j = \beta_1 \mathsf{BookVal}_j + \beta_2 \mathsf{Blemish}_j \cdot \mathsf{BookVal}_j - 2.18$$

$$\sigma_j = \beta_3 \mathsf{BookVal}_j + \beta_4 \mathsf{Blemish}_j \cdot \mathsf{BookVal}_j$$

$$log\lambda_j = \beta_5 + \beta_6 \mathsf{LogBookVal}_j + \beta_7 \mathsf{Negative}_j + \beta_8 \frac{\mathsf{MinBid}_j}{\mathsf{BookVal}_j}.$$

To check that the variable selection procedure assigns small posterior inclusion probabilities to insignificant covariates in the model, we include one superfluous covariate in each of μ_j, σ_j and λ_j, with each additional covariate drawn independently from the standard normal distribution. To speed up the computations we use the approximate bid solution in (2.6). We have checked that the exact and approximate bid function give very similar results by replicating the analysis on several simulated datasets. The empirical application on eBay coin auctions in Section 4 also provides reassuring evidence that this approximation does not distort inferences.

In Figure 3.1, posterior inclusion probabilities for each of the covariates in 50 simulated datasets are shown as box plots for different values of c. Other priors than $g = 4$ and settings with unequal values of c_μ, c_σ, and c_λ gave very similar results. As we can see in Figure 3.1, the inclusion probabilities for the most significant variables are all close to one and differ very little across the different priors. The inclusion probabilities for parameter β_4 and β_7 are low as these coefficients are quite close to zero in the generating model. Figure 3.1 also shows that insignificant variables obtain a higher posterior inclusion probability when the prior is tighter around zero, i.e. when c is small, since the cost of including an covariate with small coefficient is lower when c is small. Finally, most of the Metropolis-Hastings runs gave an acceptance probability in the range 0.6-0.8, only a few below 0.1, and none below 0.25 for the prior $c_\mu = c_\sigma = c_\lambda = n$.

4. Application to eBay auction data

4.1. **Description of the data.** Our dataset contains bid sequences and auction characteristics from 1000 eBay Internet auctions of U.S. proof sets (specially packaged collectors' coins sold by the U.S. Mint). The auctions ended between November 7 to December 19, 2007 or December 27, 2007 to January 22, 2008. We exclude multi-unit objects, auctions with a *Buy It Now* option, and Dutch auctions. Contrary to BH, we could not observe the secret reserve prices in our dataset since this information is automatically removed after the secret reserve price has been met. We also collected data from 48 additional auctions between January, 23 to January, 29, 2008 to evaluate the models' out-of-sample predictions of auction prices. Appendix D gives a detailed description of the variables in the dataset, and Table 1 presents summary statistics.

The seller's shipping cost is added to each bid, and for the auctions without listed shipping cost we use the mean of the reported shipping costs in the sample. The highest bid is typically

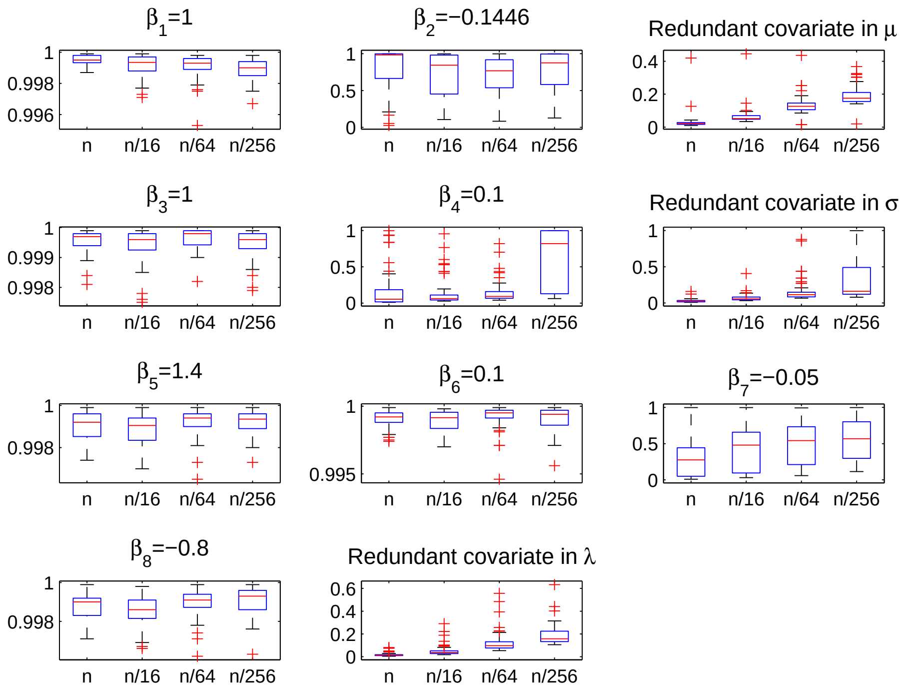

FIGURE 3.1. Posterior variable selection probabilities for the simulated data for different values of $c = c_\mu = c_\sigma = c_\lambda$ on the horizontal axis. $\bar{\kappa} = 0.25$, $g = 4$ and $\pi = 0.2$ were used in all simulations.

TABLE 1. Summary statistics of the eBay data set

Variable	Mean	Std Dev	Min	Max
Book	32.33	37.26	7.50	399.50
Price/Book, no Blemish	0.76	0.30	0.16	3.63
Price/Book, Blemish	0.61	0.24	0.05	1.20
MinBlem	0.09		0	1
MajBlem	0.04		0	1
Number of Bidders	3.74	2.70	0	15.00
MinBidShare	0.58	0.38	0	1.70
Seller's Feedback Score	2427	2883	0	12568
Power	0.53		0	1
ID	0.06		0	1
Sealed	0.10		0	1
% Sold	0.86			

NOTE: The variables are defined in Appendix D

not observed in eBay auctions, unless the difference between the two highest proxy bids is less than eBay's bid increment. The highest bid is observed in 109 of the 1000 auctions and we modify the likelihood function in (2.5) accordingly.

The bids recorded on eBay's Bid History page are supposed to correspond to the final bids for each bidder. A careful inspection of the bids reveals, however, that some bids are only a tiny fraction of the object's book value, and cannot realistically represent serious final bids. We will therefore exclude the most extreme bids in our main analysis. A bid b is excluded if

$b \leq \delta \cdot \min(\mathsf{Book},\mathsf{Price})$, where $\delta = 0.25$ in the benchmark estimations (this excludes 107 bids from the 3742 bids in the sample). We also estimate the models on all bids ($\delta = 0$) and with $\delta = 0.5$ (431 bids removed).

4.2. **Estimation results.** We perform some transformations on the covariates before the analysis to better match the functional form of the model and to reduce correlation between some of the slope coefficients and intercepts. We use the notation, $x_d = x - \bar{x}$ and $\mathrm{L}x = ln(x)$, where x is a covariate. As an example, LBook_d is the deviation of log book value from the mean log book value.

TABLE 2. Tobit regressions of auction prices censored at the minimum bid

Covariate	Mean	Stdev	Incl Prob	IF
Const	27.177	0.230	1.000	1.789
Book_d	0.757	0.007	1.000	1.796
Book·Power	−0.005	0.009	0.039	–
Book·ID	0.032	0.021	0.118	–
Book·Sealed	0.321	0.018	1.000	2.063
Book·MinBlem	0.003	0.014	0.035	–
Book·MajBlem	−0.151	0.023	1.000	1.601
Book·NegScore	0.006	0.013	0.032	–
σ_ε	6.664	0.158	–	1.938

NOTE: $x_1 \cdot x_2$ denotes the interaction of x_1 and x_2.

TABLE 3. Poisson regression with the number of bids as the response variable

Coeff	Covariate	Mean	Stdev	Incl Prob	IF
λ	Const	1.056	0.023	1.000	1.694
	Power	−0.031	0.037	0.010	–
	ID	−0.401	0.093	0.997	1.304
	Sealed	0.444	0.049	1.000	1.379
	MinBlem	−0.027	0.055	0.005	–
	MajBlem	−0.235	0.090	0.111	1.615
	NegScore	0.085	0.056	0.011	–
	LBook_d	−0.113	0.028	0.973	1.416
	$\mathrm{MinBidShare}_d$	−1.894	0.074	1.000	2.797

As a first check on the data, we ran a Tobit regression with the auction price as the response variable (censored at the minimum bid). Table 2 presents the results from the Bayesian estimation with variable selection (using the algorithm in Appendix C). An auction with an object in good condition with a large book value in an unopened envelope is very likely to sell for a large price. The seller's feedback score or status (Power or ID) does not seem to matter for prices.

Table 3 presents the results from a Poisson regression with the number of bids as the response variable. Posting a high minimum bid is clearly seen to strongly discourage entry, and the same holds to some extent for a high book value. A verified seller (ID) seems to attract less bidders, but this will turn out to not hold in the structural models.

We will now turn to the inferences from the two structural models with Gaussian and Gamma distributed values. We will mainly concentrate on the Gaussian model, however, as

TABLE 4. Posterior inferences for the eBay data from the Gaussian model with the exact and the approximate bid function

		Mean		St Dev		Incl prob	
Parameter	Covariate	Approx	Exact	Approx	Exact	Approx	Exact
κ	-	5.499	5.287	0.772	0.375	-	-
μ	Const	28.273	28.604	0.245	0.403	1.000	1.000
	Book_d	0.740	0.732	0.010	0.013	1.000	1.000
	Book·Power	0.033	0.052	0.015	0.017	0.064	0.451
	Book·ID	0.128	0.121	0.036	0.064	0.900	0.046
	Book·Sealed	0.372	0.450	0.029	0.034	1.000	1.000
	Book·MinBlem	−0.022	−0.018	0.021	0.022	0.010	0.013
	Book·MajBlem	−0.252	−0.214	0.030	0.029	1.000	1.000
	Book·NegScore	−0.003	−0.023	0.018	0.027	0.004	0.025
$log(\sigma^2)$	Const	3.997	4.309	0.071	0.051	1.000	1.000
	LBook_d	1.262	1.533	0.038	0.035	1.000	1.000
	LBook·Power	0.043	0.057	0.018	0.019	0.220	0.628
	LBook·ID	0.042	0.082	0.040	0.036	0.481	0.122
	LBook·Sealed	0.211	0.241	0.027	0.023	1.000	1.000
	LBook·MinBlem	−0.028	0.021	0.027	0.026	0.012	0.003
	LBook·MajBlem	0.036	−0.025	0.040	0.029	0.007	0.029
	LBook·NegScore	0.035	0.046	0.021	0.012	0.017	0.143
$log(\lambda)$	Const	1.193	1.269	0.021	0.022	1.000	1.000
	Power	0.009	−0.020	0.035	0.033	0.005	0.009
	ID	−0.177	−0.129	0.110	0.092	0.030	0.003
	Sealed	0.323	0.322	0.048	0.047	1.000	1.000
	MinBlem	−0.049	−0.127	0.048	0.045	0.008	0.004
	MajBlem	−0.151	-0.172	0.085	0.060	0.019	0.016
	NegScore	0.055	0.085	0.049	0.055	0.012	0.018
	LBook_d	−0.038	-0.119	0.027	0.028	0.018	0.989
	MinBidShare_d	−1.433	-1.591	0.056	0.075	1.000	1.000

NOTE: $c = n, \bar{\kappa} = 0.25, g = 4$, and $\pi = 0.2$. $x_1 \cdot x_2$ denotes the interaction of x_1 and x_2.

this model is shown to outperform the Gamma model for this particular data set, see Section 4.3. The Gaussian model is also faster to estimate and gives a higher efficiency in the MCMC posterior sampling. The Gaussian model is given implicitly by the linear equations for μ, $\ln\sigma$ and $\ln\lambda$ in Table 4. Contrary to BH we do not subtract the shipping cost in the μ-equation (the shipping cost of each auction is instead added to the bids, see Section 4.1 above).

Table 4 reports the estimation results for the Gaussian model, both for the exact and approximate bid function. The exact inverse bid function is computed by first evaluating the bid function for a single auction over a grid of signals. This discretized bid function is then linearly extrapolated to the other auctions using a polynomial fitting method, see BH for details. The results in Table 4 are based on $10,000$ MCMC draws after a burn-in of $2,000$ iterations. The MH acceptance probability in the model with the exact bid function was 53% (compared to 59% in the approximate case). The mean and maximal inefficiency factors when the exact bid function is used are 12.36 and 46.21, which should be compared to 5.32 and 10.90 in the approximate case. This means that one needs roughly $2.5 - 4$ times as many posterior draws in the exact case to obtain the same numerical precision as when the approximate bid

Table 5. Comparing the posterior inference for the eBay data from the Gaussian and Gamma models

		Mean		St Dev		Incl Prob	
Parameter	Covariate	Gauss	Gamma	Gauss	Gamma	Gauss	Gamma
κ/τ	–	5.499	2.997	0.772	0.111	1.000	1.000
μ	Const	28.273	28.307	0.245	0.304	1.000	1.000
	Book_d	0.740	0.747	0.010	0.012	1.000	1.000
	Book·Power_d	0.033	0.046	0.015	0.018	0.064	0.107
	Book·ID	0.128	0.052	0.036	0.039	0.900	0.017
	Book·Sealed	0.372	0.488	0.029	0.051	1.000	1.000
	Book·MinBlem	−0.022	0.002	0.021	0.028	0.010	0.008
	Book·MajBlem	−0.252	−0.269	0.030	0.040	1.000	1.000
	Book·LargNeg	−0.003	−0.020	0.018	0.025	0.004	0.009
$log(\sigma^2)$	Const	3.997	4.314	0.071	0.038	1.000	1.000
	LBook_d	1.262	1.276	0.038	0.026	1.000	1.000
	LBook·Power	0.043	0.069	0.018	0.020	0.220	1.000
	LBook·ID	0.042	0.032	0.040	0.067	0.481	0.011
	LBook·Sealed	0.211	0.362	0.027	0.019	1.000	1.000
	LBook·MinBlem	−0.028	−0.057	0.027	0.026	0.012	0.039
	LBook·MajBlem	0.036	0.063	0.040	0.049	0.007	0.017
	LBook·NegScore	0.035	0.042	0.021	0.027	0.017	0.050
$log(\lambda)$	Const	1.193	1.234	0.021	0.022	1.000	1.000
	Power	0.009	−0.028	0.035	0.029	0.005	0.012
	ID	−0.177	−0.197	0.110	0.078	0.030	0.048
	Sealed	0.323	0.331	0.048	0.048	1.000	1.000
	MinBlem	−0.049	−0.042	0.048	0.048	0.008	0.009
	MajBlem	−0.151	−0.115	0.085	0.097	0.019	0.015
	NegScore	0.055	0.086	0.049	0.047	0.012	0.022
	LBook_d	−0.038	−0.036	0.027	0.021	0.018	0.031
	MinBidShare_d	−1.433	−1.380	0.056	0.059	1.000	1.000

NOTE: The approximate bid function was used for both models. $c = n, \bar{\kappa} = 0.25, g = 4$, and $\pi = 0.2$. $x_1 \cdot x_2$ denotes the interaction of x_1 and x_2.

function is used. We note that an inefficiency factor equal to one is an ideal that is typically never obtained (the draws are then as efficient as iid sampling from the posterior). In many models of this size and complexity the IFs are often counted in hundreds, or sometimes even in thousands, so the IFs obtained here indicates a very efficient posterior sampling algorithm. The MCMC sampling takes roughly 12 times longer when the exact bid function is used compared to the approximate case. This means that the computing time to obtain the equivalent of an iid draw is roughly $30 - 48$ times larger in the exact case relative to the approximate case. It should be noted that the computing time can be reduced substantially in the exact case by lowering the precision in the numerical integration and by using a polynomial interpolation of lower order when computing the inverse bid functions. However, this reduction in computing time comes at the expense of a much less efficient MCMC algorithm, and may cause the MCMC chain to get stuck for long spells.

A first observation from Table 4 is that the posterior means and standard deviations from the approximate and the exact bid functions are practically identical. The estimated coefficients are mostly of the expected sign and have reasonable magnitudes. The book value plays a central role in all of μ, σ and λ. The covariate Sealed is also a very important in all parts of the model. It is interesting to note that eBay's detailed seller information seems to be of little use to buyers: the covariates Power, ID and NegScore (dummy for a large proportion of negative feedbacks from buyers) have almost invariably very small posterior inclusion probabilities (a possible exception is ID in μ). We experimented with other transformations of the negative feedback score and also transformations of the overall feedback score as substitutes for Power and ID, with unchanged results. This result is in contrast to the reduced form analysis in BH where it is found that overall reputation is a significant predictor of auction prices. Overall reputation was not used as a covariate in BH's structural model. The model in Table 4 will be our benchmark model.

Table 5 compares the inferences from the Gaussian model to the ones from the Gamma model in Section 2.3. With a couple of minor exceptions, the posterior distribution and variable selection results are strikingly similar in both models.

The posterior means and standard deviations in Table 5 are robust to a variety of different priors (e.g. $c = n/16, c = n/64$ and $c = n/256$, or changing $\pi = 0.5$); results are available from the authors by request. Similarly, the posterior inclusion probabilities are not altered enough to overturn any conclusions regarding the importance of individual covariates. The results were also essentially unchanged when we used all the bids ($\delta = 0$) or when more bids were removed ($\delta = 0.5$) compared to the benchmark setting.

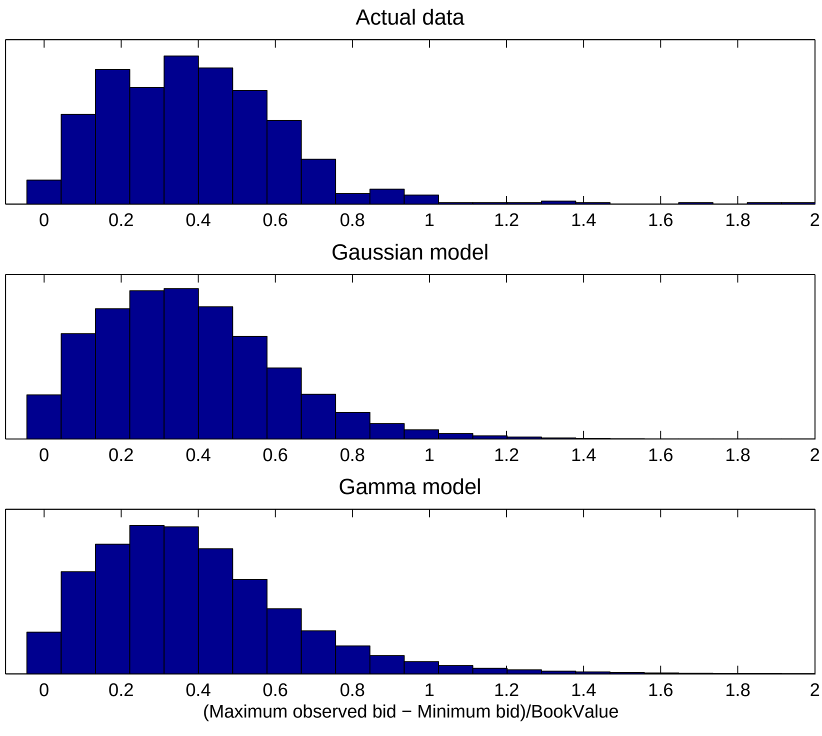

FIGURE 4.1. Posterior predictive analysis of the models' fit of the within-auction variation.

4.3. **Model evaluations and predictions.** We use posterior predictive analysis to evaluate the in-sample fit of the models, i.e. we compare the observed data to simulated data from the estimated models (Gelman et al., 2004). Given the observed auction-specific covariates,

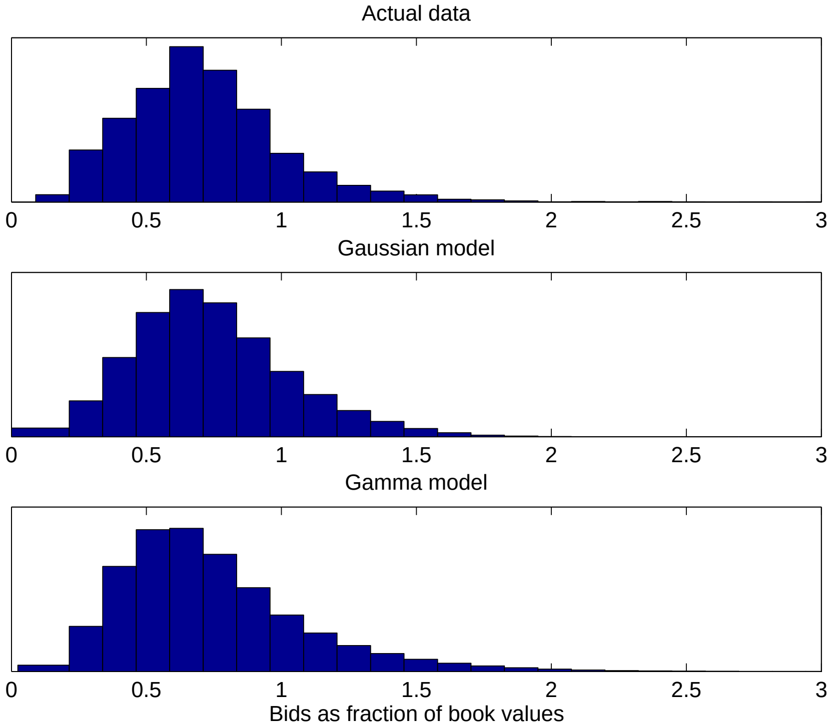

FIGURE 4.2. Posterior predictive analysis of the models' fit of the cross-auction heterogeneity.

we simulated 100 new complete datasets for each of a 100 systematically sampled posterior draws of the model parameters. This gives us $10,000$ full datasets, each with bids from a 1000 auctions.

Following BH, we compare the observed and simulated data through two summary statistics: *within-auction bid dispersion* and *cross-auction heterogeneity*. The within-auction dispersion is defined as the difference between the highest observed bid and the lowest bid divided by the auctioned item's book value, and the cross-auction heterogeneity is investigated by histograms of the bids divided by the corresponding book value in that auction. As we can see in Figures 4.1 and 4.2, the observed within-auction bid dispersion and the cross-auction heterogeneity are very well captured by both the Gaussian and the Gamma models, and the differences between the two models are small. This is in contrast to BH where the within-auction bid dispersion is under-estimated and the cross-auction heterogeneity is highly over-estimated. BH suggest adding unobserved heterogeneity to the model as a way to improve the in-sample fit of the model, but our results in Figures 4.1 and 4.2 suggest that such extensions are not needed to capture the variation in the bids.

We also evaluate the out-of-sample predictions of auction prices in a dataset with 48 additional auctions of *U.S.* proof sets. These auctions were all completed in the week directly following the estimation sample. Given the covariates from these auctions, we simulated predictive price distributions for each auction in a similar way as above. The 48 hold-out auctions were not used when computing the posterior distribution. Figure 4.3 illustrates the predictive distributions from both models for half of the out-of-sample auctions (to save space, we do not plot all auctions). Note that the predictive distributions have three components: i) a probability that the item is not sold, p_0, ii) a point mass with probability p_1 at the minimum bid (which is the final price when there is a single bidder in the auction), and iii) a continuous

Table 6. LPDS comparison of the Gaussian and Gamma models

Measure	Gaussian	Gamma
$LPDS_c$	-131.239	-136.308
$LPDS_d$	-25.248	-24.720

NOTE: $LPDS_c$ is computed on the 37 test auctions that had at least two bids, and hence a realized price above the minimum bids.

price density conditional on there being at least two bidders in the auction. The predictive price distributions look reasonable and capture the observed prices very well in most cases (note that the integral of the continuous part of the distribution equals the probability of at least two bids).

Table 6 summarizes the results from the out-of-sample predictions using the *log predictive density score* (LPDS). LPDS is the log of the predictive density evaluated at the realized price in the 48 out-of-sample auctions. Since not all auctions end with a realized price we compute two versions of the LPDS: LPDS_c evaluates the log predictive density in the 37 auctions with at least two bids, while LPDS_d is a multinomial type of evaluation that focuses on the three discrete events: i) no bids, ii) one bid (where price = minimum bid) and iii) more than one bid. LPDS_d is evaluated on all 48 out-of-sample auctions. As can be seen from Table 6, the Gaussian model outperforms the Gamma model in terms of LPDS_c by nearly 6 units on the log scale, which is very large difference given the relatively small hold-out sample. The difference between the two models in LPDS_d is negligible, however.

We also conducted a small simulation study (not reported in detail here) with bids simulated from a Gamma distribution with three different degrees of skewness: 0.5, 1 and 2. The Gamma model was only slightly better than the Gaussian model for the least skewed data, but outperformed the Gaussian model substantially in the two other cases.

4.4. **Winner's curse.** The winner's curse is by far the most highlighted phenomena in common value auctions (Thaler, 1988). In equilibrium, bidders react to the winner's curse by lowering their bids as the number of bidders increases (Krishna, 2002), see the left part of Figure 4.4 for an illustration based on the exact bid function in the Gaussian model. The Winner's curse is the net result of two opposite forces on the bids. First, more bidders leads to more competition which gives a bidder incentives to submit a higher bid (*competition effect*). This effect is more than offset by an *overestimation effect*: since the winner has the largest signal, the probability that the winner has over-estimated the object's value increases with the number of bidders. The right part of Figure 4.4 shows that the approximate bid function in the Gaussian model captures the winner's curse effect very well for signals near the center of the distribution. The linearity of the approximation has the consequence that the winner's curse effect is reversed for large enough signals, but those signals lie almost three standard deviations above the expected value and are therefore highly improbable. To investigate this more generally, we look at the first derivative of the approximate bid function with respect to n

$$b_n'(x) = \frac{\gamma\kappa\left[x - \mu - \frac{\sigma\theta(2+\kappa)}{\sqrt{\kappa}}\right]}{2\left[\gamma(n-2) + 1 + \frac{\kappa}{2}\right]^2}. \tag{4.1}$$

Since $x \sim N[\mu, (\kappa+1)\sigma^2]$, $x_d = \mu + d\sigma\sqrt{\kappa+1}$ is a signal that lies d standard deviations above the mean. It is now easy to see that $b_n'(x_d) > 0$ for all κ if $d > \frac{\theta(2+\kappa)}{\sqrt{\kappa(\kappa+1)}} > \theta = 1.96$. This

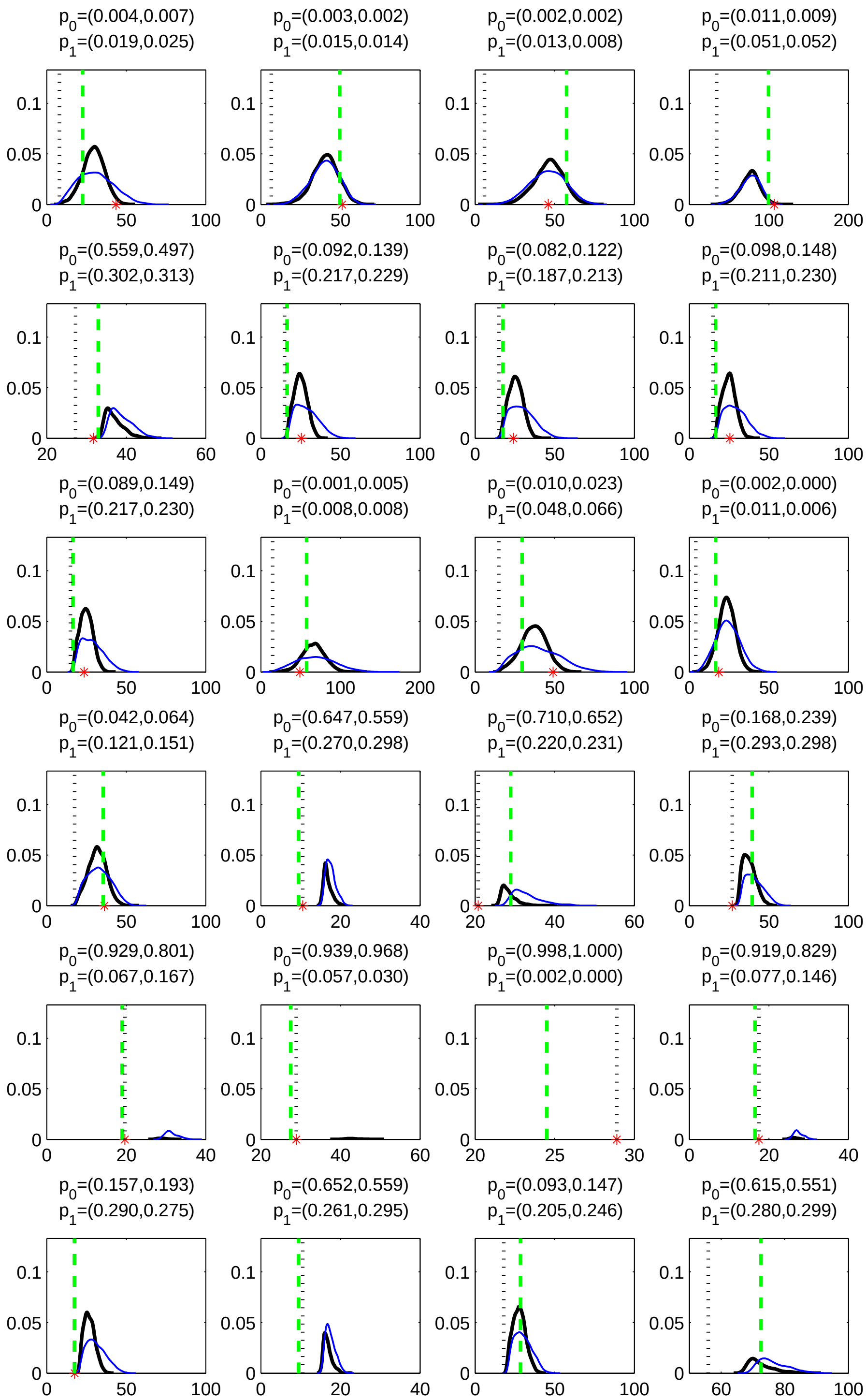

FIGURE 4.3. Predictive distributions for the auction price in 24 of the auctions in the hold-out sample. The values of p_0 and p_1 in the titles are the predictive probabilities of zero and one bid (where the price is not observed) for the Gaussian (first number) and Gamma models (second number). The densities are the predictive densities of the price when the auction has at least two bidders (the integral of the density is the probability of at least two bids) for the Gaussian (thick line) and Gamma (thin line) models. The vertical lines indicate the minimum bid (dotted) and the book value (dashed). The realized price (if observed) is displayed by the star symbol.

means that our approximation preserves the winner's curse effect for all signals up to at least 1.96 standard deviations above the mean, regardless of κ. Moreover, d is much larger than 1.96 for many values of κ that occur in practice: for example when $\kappa = 4.5$ we have $d = 2.56$, and for $\kappa = 1$ we get $d = 4.16$.

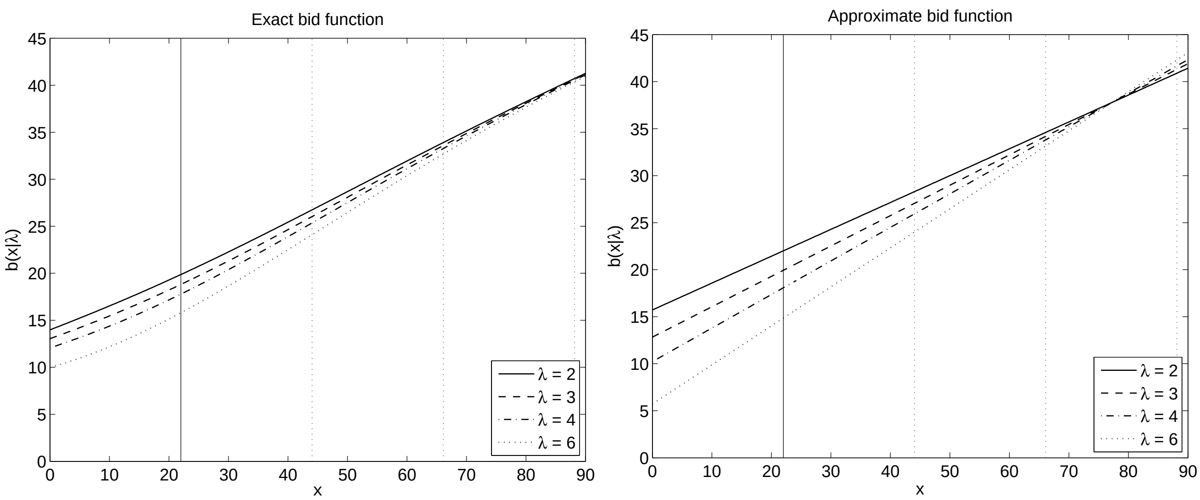

FIGURE 4.4. Illustrating the winner's curse using the exact (left) and approximate (right) bid functions.

4.5. **Determinants of expected seller revenue.** We compute the expected seller revenue by simulating from the posterior predictive distribution of the auction price, in a similar way to how we obtained the measures of in-sample fit in Section 4.3. Figure 4.5 shows that the expected seller revenue increases with n and μ (in our representative eBay auction using the Gaussian model). The difference between the results from the exact (left graph) and approximate (right graph) bid functions is small. We also found that a higher variance of the common value v decreases expected seller revenue (not shown).

The seller needs to decide on a minimum bid; a low minimum bid encourages entry, but exposes the seller to the risk of a low sale price. A risk-neutral seller will choose the minimum bid that maximizes expected revenue,

$$\mathrm{E}[\mathrm{Revenue}] = \Pr(\mathrm{Sale}|r)\mathrm{E}(\mathrm{Price}|r) + [1 - \Pr(\mathrm{Sale}|r)] \cdot \mathrm{ResidualValue},$$

where ResidualValue is the value of the object to the seller if he fails to sell it. The left side of Figure 4.6 depicts the seller's expected revenue as a function of the ratio of the minimum bid to the book value (MinBidShare) when the residual value is set to four different fractions of the mean valuation μ. The optimal minimum bid is as high as 92% of the object's book value when the residual value is μ, roughly 70% when the residual value is 80% of μ, and zero when the residual value is below or equal to 0.7μ. These are slightly higher optimal minimum bid than the ones obtained in BH, with the exception of the lowest residual value. It is typically optimal for the seller to place a minimum bid around 90-100% of the residual value, which is close to the optimal 80-90% of the residual value in BH. The right hand side of Figure 4.6 shows that the probability intervals of the number of bidders shrink as MinBidShare increases. The probability of zero entry is essentially unity beyond the point MinBidShare$= 1$.

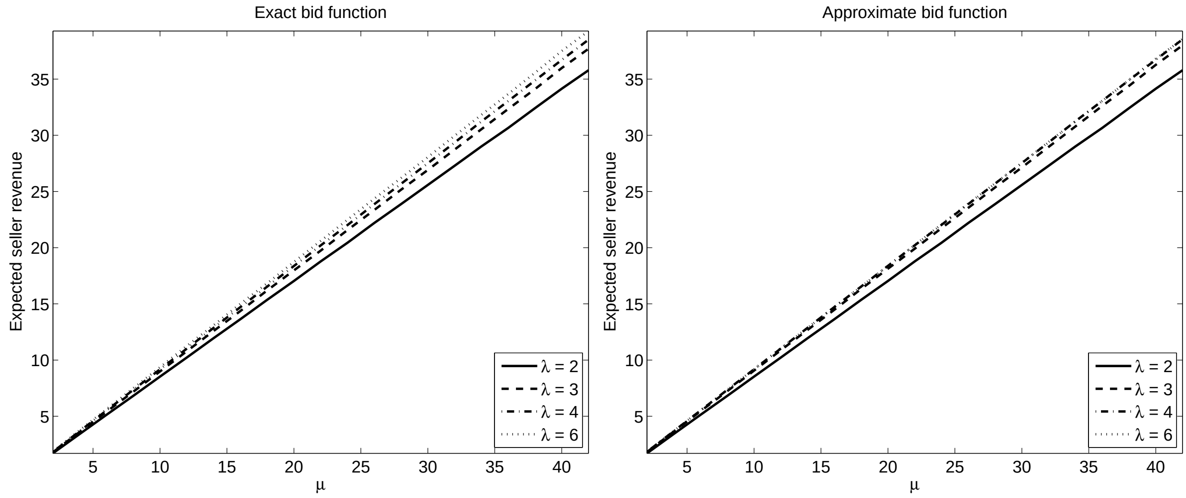

FIGURE 4.5. Expected seller revenue as a function of μ for different number of bidders using the exact (left) and approximate (right) bid function.

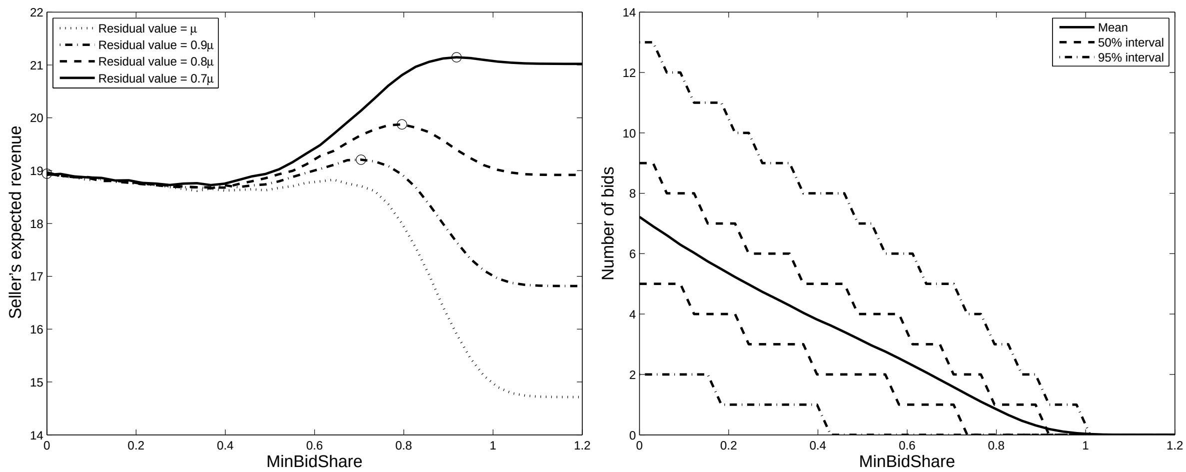

FIGURE 4.6. Analysis of the optimal choice of minimum bid. The figure shows the posterior distribution of the sellers expected revenue (left) and the number of bids, i.e. the number of bidders with signals $x > x^{\star}$, (right) as a function of the proportion of the minimum bid to the book value. The circles in the figure to the left marks out the optimal MinBidShare value for each residual value. Note that the optimal minimum bid is zero when the residual value is 0.7μ.

5. Conclusions

Our paper contributes to the technically challenging econometric analysis of second-price common value auctions. Building on the seminal work of Bajari and Hortaçsu (2003), we propose a Bayesian framework to analyze auction datasets on a routine basis using a prior distribution with a small number of easily specified prior hyperparameters. We also extend the model in Bajari and Hortaçsu (2003) to non-negative values and signals following the Gamma distribution. One of the key features of our approach is an analytical approximation of the otherwise highly complicated equilibrium bid functions. The approximate bid functions are analytically invertible and differentiable and can therefore be used for a fast and numerically stable evaluation of the likelihood function. Our proposed posterior sampling algorithm has the ability to handle variable selection among the model's covariates which makes it possible to use many covariates in the model. The algorithm is documented to be very efficient on real and simulated data.

We applied the models on a new collected dataset from coin auctions on eBay. The results pointed strongly to book values as the most important predictor of common values. Interestingly, the detailed seller information provided by eBay, and eBay's feedback score system seemed to be of very little importance to the buyers. These covariates did consistently get a low posterior inclusion probability in the estimations. The seller's minimum bid was found to strongly discourage auction entry, but a relatively high minimum bid was nevertheless found to maximize sellers expected revenue. The estimated eBay model captured the within-auction bid dispersion and the cross-auction heterogeneity very well. The predictive price distributions looked reasonable and captured the observed prices very well in most cases.

Possible extensions of the models could be to introduce elements of incremental bidding (Ockenfels and Roth, 2006) and to consider the effects of bidders' search for low-price auctions (Sailer, 2006). More generally, it would be interesting to carefully model the dependence between auctions, especially auctions with overlapping time frames. Other possible extensions include auctions with both a private and a common value element of the object, multi-unit objects, or auctions with risk-averse bidders.

Appendix A. approximating the bid function in the Gaussian model

We focus first on the case with a known number of bidders n and then generalize the results to the case with stochastic entry. The derivation of the linear approximation is divided into three steps.

Step 1. By substituting $t = \frac{x-v}{\sqrt{\kappa}\sigma}$ the bid function in (2.1) becomes

$$b(x) = x - \sqrt{\kappa}\sigma \frac{\int_{-\infty}^{\infty} te^{-t^2}\Phi^{n-2}(t)e^{-\frac{1}{2\sigma^2}(x-\sqrt{\kappa}\sigma t-\mu)^2}\,dt}{\int_{-\infty}^{\infty} e^{-t^2}\Phi^{n-2}(t)e^{-\frac{1}{2\sigma^2}(x-\sqrt{\kappa}\sigma t-\mu)^2}\,dt},$$

where $\Phi(\cdot)$ is the standard normal *c.d.f.*

Step 2. Let $h(t|\gamma,\theta) = e^{-\gamma\cdot(t-\theta)^2}$ be the approximating function to the standard normal distribution function $\Phi(t)$ on $[-a,a]$, see Figure A.1. The approximation error outside this interval is likely to have only a minor effect on the approximation of $b(x)$ as we choose a in the tails of the signal density, see below. The approximation constants γ and θ are obtained by minimizing the maximum divergence between $h(t|\gamma,\theta)$ and $\Phi(t)$ over $[-a,a]$, i.e.

$$(\hat{\gamma},\hat{\theta}) = \min_{\gamma,\theta}\left(\max_t |h(t|\gamma,\theta) - \Phi(t)|\right).$$

For $a = 2$, we obtain

$$(\hat{\gamma},\hat{\theta}) = (0.1938, 1.9600).$$

Figure A.1 shows that the strictly increasing $\Phi(t)$ is well approximated by the Gaussian density at most values over the target domain $[-2,2]$. The figure also shows that choosing $a = 1$ gives a worse approximation in the tails, and $a = 3$ does somewhat better in the tails than $a = 2$, but looses out in the main part of the distribution. The posterior inferences in the eBay's data used in Section 4 are strikingly similar for $a = 1, 2$ or 3, but the value $a = 2$ gave the best approximation of the exact bid function (results can be obtained from the authors by request). We therefore use $a = 2$ in the paper.

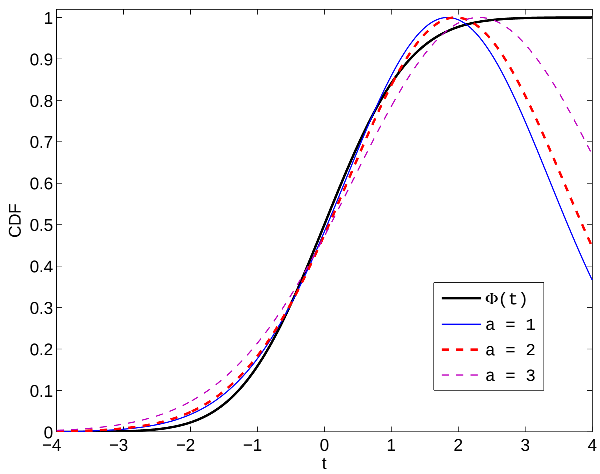

FIGURE A.1. Approximating the Gaussian CDF with a Gaussian PDF over the interval $[-a, a]$.

Step 3. Replacing $\Phi(t)$ by $h(t|\hat{\gamma}, \hat{\theta})$, the approximated bid function becomes

$$b(x) \approx x - \sqrt{\kappa} \cdot \sigma \cdot \frac{\int_{-\infty}^{\infty} t \cdot e^{-t^2} \cdot e^{-(n-2)\hat{\gamma}(t-\hat{\theta})^2} \cdot e^{-\frac{1}{2\sigma^2}\cdot(x-\sqrt{\kappa}\sigma t-\mu)^2}\, dt}{\int_{-\infty}^{\infty} e^{-t^2} \cdot e^{-(n-2)\hat{\gamma}(t-\hat{\theta})^2} \cdot e^{-\frac{1}{2\sigma^2}\cdot(x-\sqrt{\kappa}\sigma t-\mu)^2}\, dt}.$$

By completing the squares of the exponential functions, the bid function $b(x)$ can be simplified to

$$\begin{aligned} b(x) &\approx x - \sqrt{\kappa} \cdot \sigma \cdot \frac{\int_{-\infty}^{\infty} t \cdot e^{-m_1(t-m_2)^2}\, dt}{\int_{-\infty}^{\infty} e^{-m_1(t-m_2)^2}\, dt} \\ &= x - \sqrt{\kappa} \cdot \sigma \cdot \frac{\mathrm{E}(t)}{1} = x - \sqrt{\kappa} \cdot \sigma \cdot m_2, \end{aligned}$$

where $m_1 = 1 + (n-2)\hat{\gamma} + \frac{\kappa}{2}$, and $m_2 = \frac{(n-2)\hat{\gamma}\hat{\theta} + \frac{\sqrt{\kappa}(x-\mu)}{2\sigma}}{1+(n-2)\hat{\gamma}+\frac{\kappa}{2}}$. Note that the constants of the normal kernel in the numerator and the denominator cancel out. Substituting the expression for m_2 gives the approximate bid function in (2.6).

Turning to the model with stochastic entry, it is straight-forward to see that the same type of approximations can used for the minimum bid in (2.2)

$$r(x^\star, \lambda) \approx \sum_{n=1}^{\infty} p_n(\lambda) \left(c_r + \omega_r \mu + (1-\omega_r)x^\star\right),$$

which can be solved for $x^\star$ as

$$x^\star(r, \lambda) \approx \frac{r - \sum_{n=1}^{\infty}(c_r + \mu\omega_r)p_n(\lambda)}{\sum_{n=1}^{\infty}(1-\omega_r)p_n(\lambda)}.$$

The same approach can not be used to approximate the bid function in (2.1). We could do the approximation term by term in the summation, but the bid function can then no longer be inverted analytically. One way to proceed is to note that the bid function can be expressed as

$$b(x) = \frac{\mathrm{E}_{n|\lambda}\left[(n-1)g_1(n)\right]}{\mathrm{E}_{n|\lambda}\left[(n-1)g_2(n)\right]}, \tag{A.1}$$

where $g_1(n) = \int_{-\infty}^{\infty} v \cdot F_{x|v}^{n-2}(x|v) \cdot f_{x|v}^2(x|v) \cdot f_v(v)dv$, $g_2(n) = \int_{-\infty}^{\infty} F_{x|v}^{n-2}(x|v) \cdot f_{x|v}^2(x|v) \cdot f_v(v)dv$, and $\mathrm{E}_{n|\lambda}$ denotes the expectation with respect to the Poisson distribution of the number of bidders n. A first-order Taylor expansion of $(n-1)g_1(n)$ and $(n-1)g_2(n)$ around $n = \lambda$ then gives

$$b(x) = \frac{\mathrm{E}_{n|\lambda}\left[(n-1)g_1(n)\right]}{\mathrm{E}_{n|\lambda}\left[(n-1)g_2(n)\right]} \approx \frac{g_1(\lambda)}{g_2(\lambda)}, \tag{A.2}$$

where the ratio $g_1(\lambda)/g_2(\lambda)$ can now be approximated with the linear approximation in (2.6) with $n = \lambda$. Figure 2.1 and the estimation results in Section 4.2 verify that this gives a quite accurate approximation of the true bid function. The reason for this is that $g_1(n)$ and $g_2(n)$ are very similar functions and the approximation errors in the numerator and denominator in (A.2) therefore cancel out, see Tierney and Kadane (1986) for similar results in a more general setting.

Appendix B. Approximating the bid function in the Gamma model

In the Gamma model we approximate the survival function $(1 - F_{s|v}(1/x|v))$ by a Gamma probability density function over the whole interval $(0, \infty)$. By substitution the distribution function of $s|v$ becomes

$$F_{s|v}(1/x|v) = \int_0^{1/x} \frac{(\tau v)^\tau}{\Gamma(\tau)} l^{\tau-1} e^{-\tau v \cdot l}\, dl = \int_0^{\tau \cdot \frac{v}{x}} \frac{1}{\Gamma(\tau)} t^{\tau-1} e^{-t}\, dt.$$

Hence, the distribution function $F_{s|v}$ depends on the parameter τ through the support $\frac{v}{x}$. Let $h\left(\frac{v}{x}|\xi_\tau, \psi_\tau\right) = \frac{\psi_\tau^{\xi_\tau+1}}{\Gamma(\xi_\tau+1)}\left(\frac{v}{x}\right)^{\xi_\tau} \cdot e^{-\psi_\tau \cdot \frac{v}{x}}$ be the approximating Gamma *p.d.f.* to $(1 - F_{s|v}(1/x|v))$. Then, given an arbitrary τ the approximation constants ξ_τ and ψ_τ are obtained by minimizing the maximum divergence between $h\left(\frac{v}{x}|\xi_\tau, \psi_\tau\right)$ and $(1 - F_{s|v}(1/x|v))$, i.e.

$$(\hat{\xi_\tau}, \hat{\psi_\tau}) = \min_{\xi_\tau, \psi_\tau}\left(\max_{\frac{v}{x}}\left|h\left(\frac{v}{x}|\xi_\tau, \psi_\tau\right) - \left(1 - F_{s|v}(1/x|v)\right)\right|\right).$$

The approximation constants can in principle be calculated for any τ, but in practice it is more convenient to tabulate them over a grid $\mathcal{T} = (\tau_1, ..., \tau_T)$ of values for τ. Since the approximation of the bid function needs to be solved for any τ in the estimation process, we model $\{\hat{\xi_\tau}, \hat{\psi_\tau}\}_{\tau \in \mathcal{T}}$ as a multivariate regression with several functions of τ as independent variables. The fit of the regression is improved if the grid $\mathcal{T}$ is not too wide, and we will here choose a grid that covers all relevant values of τ in our datasets. The best multivariate regression model (according to adjusted R^2) is

$$\hat{\xi_\tau} = -0.1659 - 0.0159 \cdot \tau + 0.2495 \cdot \log(1 + \tau),$$

and

$$\hat{\psi_\tau} = 0.3095 - 0.0708 \cdot \tau + 1.0241 \cdot \log(1 + \tau)$$

for $0.1 \leq \tau \leq 10$ with R^2 equal to 99.5% and 99.2%, respectively.

Now, by replacing $(1 - F_{S|V}(1/x|v))$ with $h_v(v|\hat{\xi}_\tau, \hat{\psi}_\tau)$, the approximate bid function for a known number of bidders becomes

$$b(x) \approx \frac{\int_0^\infty v \cdot v^{(n-2)\hat{\xi}_\tau+2\tau+\xi-1} \cdot e^{-(\frac{1}{x}(\hat{\psi}_\tau(n-2)+2\tau)+\psi)\cdot v}\, dv}{\int_0^\infty v^{(n-2)\hat{\xi}_\tau+2\tau+\xi-1} \cdot e^{-(\frac{1}{x}(\hat{\psi}_\tau(n-2)+2\tau)+\psi)\cdot v}\, dv}.$$

$$= \frac{\int_0^\infty Gamma(v|\xi_\tau^{'}+1, \psi_\tau^{'})\, dv}{\int_0^\infty Gamma(v|\xi_\tau^{'}, \psi_\tau^{'})\, dv} = \frac{\Gamma(\xi_\tau^{'}+1)}{\psi_\tau^{'(\xi_\tau^{'}+1)}} \cdot \frac{\psi_\tau^{'\xi_\tau^{'}}}{\Gamma(\xi_\tau^{'})}$$

$$= \frac{\xi_\tau^{'}}{\psi_\tau^{'}} = \frac{\left[\xi + 2\tau + (n-2)\hat{\xi}_\tau\right] \cdot x}{\psi x + 2\tau + (n-2)\hat{\psi}_\tau}$$

where $\xi_\tau^{'} = (n-2)\hat{\xi}_\tau + 2\tau + \xi$, $\psi_\tau^{'} = \frac{1}{x}(\hat{\psi}_\tau(n-2) + 2\tau) + \psi$ and $Gamma(v|\xi_\tau^{'}, \psi_\tau^{'})$ denotes the *pdf* of the Gamma distributed variable v with parameters $\xi_\tau^{'}$, and $\psi_\tau^{'}$.

The same approach as in the Gaussian model (see Appendix A) can be used to generalize this result to the case with a stochastic number of bidders.

Appendix C. The MH algorithm with variable selection

Starting with George and McCulloch (1993) and Smith and Kohn (1996), there has been a number of algorithms that simultaneously draw the regression coefficients from the posterior and does variable selection, all in a single run of the sampler. In particular, Nott and Leonte (2004) propose an efficient algorithm for variable selection in generalized linear models (GLM). Villani et al. (2009) extend this algorithm to a general setting with the only requirement being that the gradient of the log posterior is available in closed form. The algorithm presented below is of similar form, but can be applied to any problem as long as the likelihood and prior can be evaluated numerically, and it is therefore applicable for most auction models.

We present the algorithm for a general setting where β contains all the r model parameters and D denotes the available data. Consider now setting a subset of the elements in β to zero (any other value is also possible). Let $\mathcal{J} = (j_1, ..., j_r)$ be a vector of binary indicators such that $j_i = 0$ iff the ith element of β is zero. We shall here for simplicity assume that the elements of $\mathcal{J}$ are independent a priori with $\Pr(j_i) = \pi$ for all i, so that π is the prior probability of including the ith covariate in the model. The following algorithm samples β and $\mathcal{J}$ simultaneously using an extended Metropolis-Hastings algorithm with a proposal density of the form

$$f(\beta_p, \mathcal{J}_p|\beta_c, \mathcal{J}_c) = g(\beta_p|\mathcal{J}_p, \beta_c)h(\mathcal{J}_p|\beta_c, \mathcal{J}_c),$$

where β_p and $\mathcal{J}_p$ are the proposed values for β and $\mathcal{J}$, β_c and $\mathcal{J}_c$ are the current values for β and $\mathcal{J}$, h is the proposal distribution for $\mathcal{J}$, and g is the proposal density for β conditional on $\mathcal{J}_p$. The Metropolis-Hastings acceptance probability then becomes

$$a[(\beta_c, \mathcal{J}_c) \to (\beta_p, \mathcal{J}_p)] = \min\left(1, \frac{p(D|\beta_p, \mathcal{J}_p)p(\beta_p|\mathcal{J}_p)p(\mathcal{J}_p)/g(\beta_p|\mathcal{J}_p, \beta_c)h(\mathcal{J}_p|\beta_c, \mathcal{J}_c)}{p(D|\beta_c, \mathcal{J}_c)p(\beta_c|\mathcal{J}_c)p(\mathcal{J}_c)/g(\beta_c|\mathcal{J}_c, \beta_p)h(\mathcal{J}_c|\beta_p, \mathcal{J}_p)}\right),$$

where $p(D|\beta_p, \mathcal{J}_p)$ is the likelihood of the observed data conditional on β with zeros given by $\mathcal{J}_p$, $p(\beta|\mathcal{J})$ is the prior of the non-zero elements of β, and $p(\mathcal{J})$ is the prior probability of $\mathcal{J}$.

In any given iteration of the algorithm, we propose $\mathcal{J}_p$ by going through all β-coefficients one by one and with probability 0.2 change the indicator of the corresponding covariate (i.e. if currently $j = 1$, then we propose $j = 0$ and vice versa). Note also that the acceptance probability for these updates simplifies to

$$a[(\beta_c, \mathcal{J}_c) \to (\beta_p, \mathcal{J}_p)] = \min\left(1, \frac{p(D|\beta_p, \mathcal{J}_p)p(\beta_p|\mathcal{J}_p)p(\mathcal{J}_p)/g(\beta_p|\mathcal{J}_p, \beta_c)}{p(D|\beta_c, \mathcal{J}_c)p(\beta_c|\mathcal{J}_c)p(\mathcal{J}_c)/g(\beta_c|\mathcal{J}_c, \beta_p)}\right). \tag{C.1}$$

One can also consider an additional type of move by randomly picking a pair of covariates (one currently in the model and the other currently not in the model) and propose a switch of their corresponding indicators. More sophisticated ways to propose are also easily implemented, e.g. the adaptive scheme in Nott and Kohn (2005), where the history of $\mathcal{J}$-draws is used to adaptively build up a proposal for each indicator.

The proposal density $g(\beta_p|\mathcal{J}_p, \beta_c)$ is obtained as follows. First, in a model with all covariates included, we approximate the posterior with the $t(\hat{\beta}, -H^{-1}, h)$ density described in Section 3.2 for the case without variable selection. We then propose $\beta_p|\mathcal{J}_p$ from this multivariate t distribution conditional on the zero restrictions dictated by $\mathcal{J}_p$. Assume for notational simplicity that the elements in β has been rearranged so that $\beta = (\beta_0', \beta_p')'$ where β_0 are the p_0 zero-restricted elements of β under $\mathcal{J}_p$, and β_p are the non-zero parameters. Decompose $\hat{\beta}$ and $P = -H^{-1}$ conformably with the decomposition of β as

$$\begin{aligned} \hat{\beta} &= (\hat{\beta}_0', \hat{\beta}_p')' \\ \hat{P} &= \begin{pmatrix} P_{00} & P_{0p} \\ P_{p0} & P_{pp} \end{pmatrix}. \end{aligned}$$

Using result from the conditional distributions of subsets of multivariate-t variables (see e.g. Bauwens, Lubrano and Richard (1999, Theorem A.16)), we can now propose β_p conditional on $\beta_0 = 0$ from

$$\beta_p|(\beta_0 = 0) \sim t[\hat{\beta}_p + P_{pp}^{-1} P_{p0} \hat{\beta}_0, P_{pp}, c(\beta_0), v + p_0].$$
$$c(\beta_0) = 1 + \hat{\beta}_0'(\hat{P}_{00} - \hat{P}_{0p}\hat{P}_{pp}^{-1}\hat{P}_{p0})\hat{\beta}_0,$$

using the parametrization of the t distribution in Bauwens et al. (1999). A similar algorithm has recently been suggested by Giordani and Kohn (2010) in their adaptive sampling framework. They propose to use a mixture of multivariate normals as a proposal density rather than a multivariate t.

Appendix D. Description of eBay covariates

- **Book**. The price of the item as reported by the Internet coin seller *Golden Eagle Coins* at http://www.goldeneaglecoin.com.
- **MinBlem**. Dummy variable, coded as 1 if the proof set had a minor damage on the box or packaging according to a subjective assessment of the item using the seller's description and pictures of the auctioned object.
- **MajBlem**: Dummy variable, coded as 1 if at least one coin were missing in the package or if other major imperfections were present.
- **Power**. Dummy variable, coded to be 1 if the seller is ranked among the most successful sellers in terms of product sales and customer satisfaction on eBay.
- **ID**. Dummy variable, coded to be 1 if the seller's identity has been established by cross-checking his contact information in consumer and business databases.
- **Sealed**. Dummy variable, coded to be 1 if the proof set is sealed in its original envelope.
- **NegScore**. Dummy variable, coded to be 1 if more than 1% of the seller's feedback score from buyers have been negative.
- **MinBidShare**. Seller's reservation price as a fraction of the object's book value.

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