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The Monetary Policy Decision-Making Process and the Term Structure of Interest Rates

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Abstract

This paper presents a theoretical model of the term structure of interest rates based on the monetary policy decision-making process at modern central banks. Evaluations of explicit expressions for the spot and forward rate curve render several important results: (i) Spot and forward rates are explicit functions of the number of policy meetings during the time to maturity rather than the time to maturity itself. Consequently, the forward rate curve is step-shaped. (ii) In addition, there are calendar time effects, i.e. the position within the policy cycle is also of importance, especially for short term interest rates. (iii) The forward rate curve exhibits hump-shaped responses to economic shocks and a modified version of the Nelson-Siegel model can be obtained as a special case.

Keywords: *The term structure of interest rates, interest rate stepping, policy gap, calendar time effects, hump-shaped responses*

JEL Classification: G12, E43, E52

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1. Introduction

There appears to be a substantial discrepancy between how practitioners and model builders analyse term structure movements. On the one hand, practitioners such as central bank watchers are very aware of the fact that policy rates of central banks are normally only adjusted at certain pre-announced dates and that the adjustments are limited to a fixed size (typically 25 basis points) or a small multiple thereof. In other words, they ask questions such as: Will the Fed's funds rate be cut at the coming meeting or will that decision be postponed to a subsequent meeting? Will the cut be 25 or 50 basis points? It can be said that the judgements of the practitioners are well adapted to the actual decision-making process at central banks. On the other hand, the term structure models developed by academics and used by central banks and other financial institutions are not well adapted to the decision-making process at central banks. In these models, the dynamics of the short term interest rates (which is the most important aspect when building term structure models) typically follow an autoregressive process, where the changes are often normally distributed.

The aim of this paper is to reduce the discrepancy among practitioners and model builders by developing a model of the term structure of interest rates based on modern asset pricing theory that incorporates the major features of the monetary policy decision-making process. The main contribution of the paper is that it succeeds in meeting this aim. Moreover, the derived expressions for the spot and forward rates are explicit as well as relatively simple and several important features, such as calendar time effects and step-shaped forward rate curves, are easily demonstrated. Finally, the model is easily extended to include more economic factors.

The paper is organized in the following way. In section 2, some stylized features of term structure dynamics, many of them directly related to the monetary policy decision-making process, are presented followed by a brief review of popular term structure models with a focus on how well they capture the stylized features. Section 3 presents the model and demonstrates its most important properties. Some straightforward extensions of the model are also provided. A summary and conclusions as well as suggestions for future research can be found in section 4. An appendix contains the derivation of the model and some technical details.

2. Some stylized features and some popular interest rate models

There are some stylized features of the term structure of the interest rate that (to varying degrees) are related to the monetary policy decision-making process.

(i) Short term interests are often stable while there can be substantial fluctuations in long term interest rates. In particular, long term interest rates do exhibit high-frequency fluctuations whereas the policy rate of a central bank is normally constant between scheduled policy meetings. Models where the dynamics in the long term interest rates are driven by fluctuations in the short term interest rate will have severe problems in this context.

(ii) Short term interest rates are normally adjusted upwards or downwards by a fixed size (typically 25 basis points) or a multiple thereof (e.g. ± 50 basis points) at pre-announced dates. This pattern has been pronounced in recent years. In the early 1990's, small adjustments (< 25 basis points) that were not announced occurred quite frequently.

(iii) The decision not to adjust the policy rate is common. For instance, Gerlach-Kristen (2004) reports that during the period January 2000 to December 2003, most central banks adjusted the policy rate at less than 50 percent of the scheduled policy meetings.[1]

(vi) Strong serial correlation in changes of short term interest rates and gradualism.
Another observation is that an adjustment of the policy rate very often exhibits the same sign (and size) as the previous policy rate adjustment. This observation seems to be related to gradualism, i.e. central banks appear to adjust their policy rate in relatively small steps towards some target level.

(v) Hump shaped responses to the yield curve from economic shocks. In some sense, this feature seems to be an implication of feature (i) given that the impact of shocks is small on (sufficiently) long interest rates. However, the impact is also small for short term interest rates that mature

[1] See table 1 in Gerlach-Kristen (2005), where the policy rate decision behaviour of six central banks (The Fed, ECB, Bank of Canada, Reserve Bank of Australia, Bank of England, and Sveriges Riksbank) are displayed. Only Bank of Canada changed the policy rate at more than 50 percent of the policy meetings during the period examined.

after the following policy meeting and the maximum effect is often to be found on medium term interest rates, see e.g. Fleming and Remolona (1999).[2]

(vi) An unexpected decision not to the change the policy rate has a substantial impact on short term interest rates. At first sight, this observation appears to be natural and the issue of if it is of any importance whether a monetary policy shock emerges from an unexpected policy rate adjustment or an unexpected decision not to adjust the policy rate has not been analysed in the literature to any large extent.[3] However, some challenging problems arise when we confront term structure models with these types of events on high frequency (daily) data. The main problem is that the unexpected movements in the yield curve that are caused by an unexpected decision not to adjust the policy rate are hard to relate to movements in factors used in term structure models.[4]

A natural question is then to what extent existing term structure models capture features (i)-(vi) above. If we first look at the first generation of so-called affine models[5], such as Vasicek (1977), Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992), we find that it is hard to reconcile those models with any of features (i)-(vi). The main reason for this is that the instantaneous risk-free rate, which should be closely related to the policy rate of central banks, in these models follows a continuous time autoregressive process that does not resemble the policy rate process observed nowadays. Moreover, this kind of interest rate dynamics typically implies that shocks will have an impact on the yield curve that declines with time to maturity and that hump-shaped responses to shocks do not appear naturally in these models.[6] Of course, these models also have some merits. First, they are consistent arbitrage-free models where explicit expressions for term premia can be derived. They seem to do a relatively better job of describing the dynamics of long term interest rates and, in this context, they have been useful for analysing interest-rate-derivative securities.[7] It should also be recalled that the first generation of affine models was developed in a period when the

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[2] Fleming and Remolona (1999) report that economic announcements have the largest impact on intermediate maturities of one to five years.

[3] Andersson, Dillén and Sellin (2006) do analyse the surprising decision not to change the policy rate.

[4] However, in the case of an unexpected non-zero adjustment of the policy rate, the unexpected adjustment itself should be the main shock explaining unexpected yield curve movements.

[5] An affine model has the property that the zero coupon bond price, $B(t,T)$, can be written as $B(t,T) = \exp(a(t,T) + b(t,T)x(t))$, where $x(t)$ is a state vector and a and b are deterministic functions.

[6] In multifactor versions of these models, it is possible to obtain hump-shaped responses by letting two of the factors counterbalance each other on the short end of the yield curve.

[7] Se e.g. Hull and White (1990).

discrepancy between the actual policy rate dynamics and the interest dynamics assumed in the models was smaller than today.[8]

In recent years, affine models based on the so-called *level*, *slope* and *curvature* factors have gained some attention. These models (henceforth called LSC-models) are typically closely related to the three largest principal components to which a time series of a representative vector of forward rates or yields give rises, see e.g. Cochrane and Piazzesi (2005, 2006). Another example, an LSC-model, is provided by Diebold and Li (2006), who take the departure from a factor interpretation of the parameters in the Nelson and Siegel (1987) parameterization of the forward rate curve. These models can, more or less by construction, reproduce a substantial part of the fluctuations in the yield curve and they seem to produce good forecasts in comparison to competing models, see Diebold and Li (2006). As compared to the first generation of affine models, the LSC-models appear to be easier to reconcile with features (i)-(vi). The inclusion of a curvature factor can be seen as a way of incorporating feature (v) and a priori, there is no conflict with the other dynamic features, since the LSC-models are not derived from any specific factor dynamics. The latter observation is naturally also a problem since there is a risk that the models suffer from the inconsistency problems addressed by Björk and Christenssen (1999). The VAR-factor dynamics (that are imposed after the formulation of the specific affine model) are not likely to be consistent with the shape of the spot forward rate curves implied by the affine model. However, Christensen, Diebold and Rudebusch (2007) develop a slight modification of the dynamic Nelson Siegel model that is arbitrage-free and therefore, consistent. It is also difficult to understand what economic mechanisms cause term structure movements in LSC-models as they do not normally include macroeconomic (or other economic) factors.[9]

Examples of affine models with macroeconomic state variables can be found in the macro-finance literature, see e.g. Ang and Piazzesi (2003), Rudebusch and Wu (2004), Bekaert, Cho and Moreno (2006), and Hördahl, Tristani and Vestin (2006). These models often combine policy rules analysed in monetary analysis with no-arbitrage pricing principles known from

[8] For instance, before 1994, the Fed announced its target adjustments more randomly over time and the size of the adjustments were (sometimes large) multiples of 6.25 basis points. See Piazzesi (2005) for a further discussion of the Fed's operating procedures before and after 1994.
[9] One exception is Diebold, Rudebusch and Boragan Aruoba (2006), who add three macroeconomic factors to the LSC-factors in a VAR-model.

finance. An obvious merit of these models is that they reflect the fact that central banks react to macroeconomic developments. Moreover, in some of these models, the macroeconomic variables are treated as endogenous variables, which makes it possible to do policy analysis.[10] An important issue is how to interpret discrete time policy rules analysed in macroeconomic studies in a continuous time framework used in the finance literature. For instance, it is not very clear how important concepts from the policy rule analysis, such as gradualism or interest rate smoothing, translate into a continuous time model. Moreover, even though these papers focus on central bank behaviour, the dynamics of the policy rate are not in accordance with features (i)-(iii).

There are a few affine term structure models that try to incorporate some of the key features of the policy rate process discussed above, e.g. Farnsworth and Bass (2003) and, in particular, the interesting model of Piazzesi (2005). Indeed, the Piazzesi (2005) model can be reconciled with all features (i)-(vi). However, this model has some specification problems and it is also rather complex, which makes it hard to arrive at an explicit characterization and grasp the main implications of the model. These issues will be addressed in the next section where the main differences between the Piazzesi (2005) model and the model presented in this paper will be discussed.

Finally, theoretical models that focus on some of the key features such as (ii) and (iii) can be found in the interest stepping literature, see e.g. Eijffinger, Schaling, and Verhagen (1999), Gerlach-Kristen (2004), and Guthrie and Wright (2004), and the empirical analysis of these features includes Eichengreen, Watson, and Grossman (1985) and Gerlach (2005). However, these papers only deal with the dynamics of the policy rate and do not incorporate an analysis of the term structure of interest rates.

3. The model

The term structure model is presented in this section. We start by introducing the concept of the target interest rate.

[10] Term structure models in the finance literature describe the yield curve as a function of exogenous variables. Thus, policy analyses such as the analysis of shocks to the policy rate or the analysis of changes in policy parameters (e.g. coefficients in Taylor rules), are not meaningful in such models.

The target interest rates

We assume there to be an economic state variable, $x\,(t)$, (henceforth called the *target interest rate*) that follows an Ornstein-Uhlenbeck process according to

$$dx(t) = \kappa[\,x^* - x(t)\,]\,dt + \sigma dW(t), \tag{1}$$

where x^*, κ and σ are parameters and $dW(t)$ is an increment of a standard Wiener process. We notice that the future value of the state variable, $x(t+u)$, conditional on the time t observation, $x(t)$, is a random variable with mean $m(u, x(t))$ and variance $v^2(u)$, given by

$$m(u, x(t)) = x^* + e^{-\kappa u}\,(x(t) - x^*), \quad v^2(u) = \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa u}). \tag{2}$$

The phrase and concept target of the interest rate is taken from Gerlach (2005) who estimates the policy rate process of ECB with an ordered probit model. The actual process for the short term interest rates will inherit important properties from the target interest rate process, but exhibit short dynamics in the short run that differ from (2) due to the decision process of the central bank, as we will see below.

The dynamics of the policy rate

A central bank takes decisions on its instrument (the policy rate), $r(t)$, at dates $t_n = t_0 + ns$, where s is the constant length of the time period between policy meetings. Since we will measure time in years, a typical value of s is about 0.1. The policy rate, r, will be interpreted as an instantaneous risk-free rate. The case when investors face a risk-free rate that may differ somewhat from the policy rate is dealt with later. The actual policy rate will normally deviate from the target interest rate, which will give rise to a *policy gap*, $z(t)$, defined as

$$z(t) \equiv x(t) - r(t) \tag{3}$$

However, the interest rate decision will tend to close the policy gap over time in the sense that the probability of an upward (downward) adjustment of the policy rate is high if the policy gap is positive (negative) and large. More specifically, we assume that at time t_{n+1}, the central bank sets the policy rate according to

$$r(t_{n+1}) = r(t_n) + d[N^+_{t_{n+1}} - N^-_{t_{n+1}}], \tag{4}$$

where d is the size of a typical change in the policy rate (e.g. 25 basis points) and $N^+_{t_{n+1}}$ and $N^-_{t_{n+1}}$ are Poisson distributed random variables according to

$$N^+_{t_{n+1}} \sim \mathrm{Po}(\lambda^+(z(t_{n+1-}))), \quad N^-_{t_{n+1}} \sim \mathrm{Po}(\lambda^-(z(t_{n+1-}))) \tag{5}$$

$$\lambda^+(z(t_{n+1-})) = \lambda_0 + \beta \max[z(t_{n+1-}), 0], \quad \lambda^-(z(t_{n+1-})) = \lambda_0 - \beta \min[z(t_{n+1-}), 0], \tag{6}$$

where λ_0 is a non-negative parameter and β is a strictly positive parameter that reflects how aggressively the central bank reacts to the policy gap. The higher the value of β, the more likely it is that the central bank will adjust its policy rate. The notation $w(t_{n+1-})$ refers to the value of variable *w just before* the policy meeting at time t_{n+1} whereas $w(t_{n+1})$ refers to the value of variable *w just after* the policy meeting, i.e. the policy rate decision is included in the information set at time t_{n+1}. In particular, we have

$$z(t_{n+1-}) = x(t_{n+1}) - r(t_{n+1-}) = x(t_{n+1}) - r(t_n). \tag{7}$$

Several comments can be made. Notice first that although the Poisson parameters are non-linear in the policy gap *z*, it is easily shown that the expected change in the policy rate at the policy date is linear in *z* and of the form $E_{t_{k-}}[\Delta r(t_k)] = d\beta z(t_{k-})$. Furthermore, stochastic changes in the state variable, *x*, will alter the expectations of future monetary policy and thus affect long term yields. Thus, between policy dates, economic shocks (shocks to *x*) will lead to fluctuations in long term interest rates while short term interest rates that mature before the next policy meeting will be unaffected. Moreover, policy rate changes are a multiple of a fixed step size, (d). Finally, there is a positive probability that the decision is to leave the policy rate unchanged. In other words, the stylized features (i), (ii) and (iii) discussed in the previous section are built into the model by construction.

The model is many respects similar to the Piazzesi (2005) model, but there are at least three important differences. (i) One difference is that Piazzesi assumes adjustment of the official rate to take place randomly over time according to a Poisson process with state-dependent intensities[11], whereas I assume that the official rate can only be adjusted at fixed dates according to (4). (ii) Another difference is that the Poisson parameters in (6) are always non-negative, whereas negative Poisson intensities are possible in Piazzesi (2005), which leads to conceptual problems. Moreover, in Piazzesi (2005), fluctuations in the state variables will have a double effect in the sense that an increased intensity for an upward adjustment of the policy rate due to changes in state variables will be accompanied by a decreased intensity of a downward adjustment of the same size.[12] In contrast, in my model fluctuations in a *positive* policy gap will only affect the probability of an upward adjustment of the policy rate while the probability of a downward adjustment remains unaffected at a low level. The non-linear relationship between adjustments of the policy rate state variable assumed in this paper appears to be more realistic, but in the end it is an empirical issue. (iii) The third and probably the most important difference is that Piazzesi (2005) assumes a rather complicated model, where the adjustment probabilities are related to four state variables. This paper assumes a parsimonious and natural relationship between the policy gap (a single variable) and adjustment probabilities of the policy rate. As a consequence, very explicit term structure expressions can be derived.

Pricing formulas

Given the above set-up, an arbitrage-free price, $B(t,T)$, of a zero coupon bond at time t promising its owner a unit nominal payment at T is given by

[11] More precisely, Piazzesi (2005) assumes that during some time intervals, in which FOMC meetings are held, the intensity of an upward (downward) adjustment is $\overline{\lambda} + \lambda_X^T(X_t - \overline{X})$ ($\overline{\lambda} - \lambda_X^T(X_t - \overline{X})$), where X_t is the state vector, $\overline{X}$ its mean, λ_X^T is a row vector describing how sensitive the intensity is to fluctuations in the state variables, and $\overline{\lambda}$ is a positive scalar. Outside the policy meeting, the intensities for upward and downward adjustments are set to a common constant value $\overline{\lambda}$.

[12] The same effect will occur in the present model if the Poisson parameters are modelled in a linear fashion as $\lambda^+(z(t_{n+1-})) = \lambda_0 + \beta\, z(t_{n+1-})$ and $\lambda^-(z(t_{n+1-})) = \lambda_0 - \beta z(t_{n+1-})$. In this case, the expected change in the policy rate becomes $E_{t_{k-}}\left[\Delta r(t_k)\right] = 2d\beta z(t_{k-})$, which corresponds to eq. (7) in Piazzesi (2005).

$$B(t,T) = E_t\left[e^{-\int_t^T r(u)du}\right], \qquad (8)$$

where E_t denotes expectations conditional on information known at time t. Bond pricing formula (8) should be considered as a special case for which market prices of risks are zero. We make this simplifying assumption because we want to focus on the role of monetary policy expectations (i.e. expectations of future values of the policy rate) rather than on term premia, which are essentially determined by the market price of risks.[13]

It is convenient to introduce the notation $B_n(t,T)$, where subscript n provides us with the information that n policy meetings will take place during $[t,T]$. Exact closed form bond price formulas based on (8) are not available. However, an excellent approximate closed form expression is available.[14] Using notations $l(t) = t_1 - t$ and $\tau = (T\text{-}t_n)$ for *time to the next policy meeting* and *time between the maturity date and the last policy meeting date (during the life of the bond)*, respectively, it is shown in the appendix that the approximate bond price takes the form

$$B_n(t,T) = e^{-J_n(t,T)+c_n(t,T)}, \qquad (9)$$

where $J_n(t,T)$ is the expected integral of the policy path

$$J_n(t,T) \equiv E_t\left[\int_t^T r(u)du\right] = \varphi_n(l(t),\tau)\; r(t_0) + \eta_n(\tau)\, x^* + \pi_n(l(t),\tau)\,(x(t)\text{-}x^*), \qquad (10)$$

and $c_n(l(t),\tau)$ is a deterministic function recursively defined as

$$c_0(l(t),\tau) = 0 \qquad (11)$$
$$c_1(l(t),\tau) = 2\lambda_0(\cosh(d\tau)-1) + (1-w)^2\tau^2\nu^2(l(t))/2 \qquad (12)$$
$$c_{n+1}(l(t),\tau) = c_n(s,\tau) + \Lambda_n(l(t)),\; n = 1,2,\ldots \qquad (13)$$

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[13] It is straightforward to generalize (8) to $B(t,T) = E_t\left[e^{-\int_t^T r(u)du - \frac{1}{2}\int_t^T q^2(u)du + \int_t^T q(u)dW(u)}\right]$, where q is the market price of risk. If the market price of risk takes the (general) form $q = q_0 + \rho x$, we can replace κ and x^* with $\kappa_q = \kappa - \sigma\rho$, and $x_q^* = (\kappa x^* + \sigma q_0)/\kappa_q$, respectively, in the valuation formulas, provided that $\kappa_q > 0$. Like Piazzesi (2005), we also disregard the market price of policy risk (surprising policy rate movements).

[14] The approximation error on yields is less than 0.5 basis points for reasonable parameter values and thus, not noticeable in practice; see Appendix 4 for details.

$$\Lambda_n(u) = 2\lambda_0(\cosh(d\varphi_n(s,\tau)) - 1) + \left((1-\omega)\varphi_n(s,\tau) + \pi_n(s,\tau)\right)^2 \nu^2(u)/2, \tag{14}$$

where $\cosh(u) = 0.5[e^u + e^{-u}]$, $v^2(u)$ is given by (2) and functions φ_n, η_n, and π_n are defined as

$$\varphi_n(l(t),\tau) = l(t) + s\frac{\omega(1-\omega^{n-1})}{1-\omega} + \tau\omega^n, \tag{15}$$

$$\eta_n(\tau) = s[(n\text{-}1) - \frac{\omega(1-\omega^{n-1})}{1-\omega}] + \tau(1\text{-}\omega^n), \tag{16}$$

$$\pi_n(l(t),\tau) = \theta(l(t))[s\omega\left(\frac{1-(\omega/\gamma)^{n-1}}{\gamma-\omega} - \frac{(1-\omega^{n-1})}{1-\omega}\right) + \tau((\omega/\gamma)^n - \omega^n)], \tag{17}$$

and where

$$\theta(l(t)) = \frac{(1-\omega)}{(1-\gamma)}e^{-\kappa(l(t)-s)} \; . \; \omega = 1\text{-}\beta d, \quad \gamma = \omega e^{\kappa s}, \tag{18}$$

provided that $\gamma \neq 1$. Before we take a closer look at forward and yield curves, some remarks are in order. The bond pricing formula is affine i.e. the log bond price can be written as a linear function of the state variable $x(t)$ as $\ln B(t,T) = a(t,T) + b(t,T)x(t)$, where a and b are deterministic functions. However, in contrasts to most affine models (Piazzesi (2005) being an exception), functions a and b cannot be written as a function of time to maturity (T-t) only since they also depend on calendar time (t) separately. More precisely, for a fixed time to maturity, (T-t), bond prices mainly depend on the number of policy meetings during the life of the bond which, in turn, depends on where in the policy cycle (e.g. the time to the next policy meeting $l(t)$) we observe prices. We will examine this important feature more closely shortly. Finally, by inserting reasonable parameter values (e.g. from table 1 below) in the expressions for the convexity terms, it is easily verified that these terms are negligible and we will not analyse them in detail in what follows.

Examination of the forward rate curve.

We will now take a look at the forward rate curve implied by the bond pricing formula above. For that purpose, we sometimes use the parameter values reported in table 1 to get a feeling for some quantitative effects. The parameter values for d, s and x^* appear to be quite reasonable, even though the specific choice of parameter values may reflect features of specific central banks.[15] The chosen value of λ_0, which is broadly in accordance with findings reported by Piazzesi (2005), implies that an unchanged policy rate is the most likely decision when the policy gap is close to zero.[16] It is difficult a priori to have any beliefs about reasonable values for parameters β, κ and σ. However, as will be demonstrated shortly, it is possible to use recent empirical studies of the Nelson-Siegel factor model (see Diebold, Rudebusch, and Boragan Aruoba (2006)) to obtain rough estimates of these parameters. Moreover, forward and spot interest rates will turn out to be quite insensitive to parameter σ.

Table 1. Parameter values used in examples

D	s	β	λ_0	x^*	κ	σ
0.0025	0.125	40	0.2	0.045	0.9	0.033

We are now ready to take a closer look at the forward rate curve. The forward interest rate, $f(t,T)$, is defined as $f(t,T) \equiv -\partial \ln(B(t,T))/\partial T$,which, together with (9), yields

$$f(t,T) = \omega^{n(t,T)} r(t_0) + (1-\omega^{n(t,T)}) x^* + \theta(l(t))(e^{-\kappa s n(t,T)} - \omega^{n(t,T)})\,(x(t)-x^*) - c_n^f\,, \tag{19}$$

$$c_0^f = 0, \tag{20}$$

$$c_1^f = 2d\sinh(d\tau) + \tau(1-\omega)^2 v^2(l(t)), \tag{21}$$

$$c_n^f = \partial \ln(c_n(l(t),\tau))/\partial\tau\,,\ n = 2,3,\ldots, \tag{22}$$

where $\sinh(u) = 0.5[e^u - e^{-u}]$ and the function $v^2(\cdot)$ is given by (2). Let us take a closer look at expression (19). Let us begin by making some natural observations, such that the forward interest rate, $f(t,T)$, must take the same value for all maturities for which the number of policy meetings before maturity, $n(t,T)$, is the same. Thus, the forward rate curve will be step-shaped. Furthermore, the expression for the forward rate is surprisingly simple and essentially a weighted average of the current policy rate ($r(t_0)$), the long run value of the target interest rate (x^*) and the target interest gap ($x(t)$-x^*), where the weights depend on the number of policy meetings ($n(t,T)$) during $[t,T]$. In particular, the forward rate equals the current policy

[15] For instance, the chosen value of s implies eight policy meetings per year, which is the meeting frequency of the Fed. ECB, on the other hand, has 11 policy meetings per year, whereas Sveriges Riksbank recently announced that there will be six policy meetings per year.

[16] Piazzesi (2005) reports an estimate of 84 for the constant intensity $\bar{\lambda}$ (in the constant volatility case), which corresponds to a value of 84/365 = 0.23 for λ_0.

rate, $r(t_0)$, when there are no policy meetings before maturity ($n(t,T)$ = 0), whereas it tends to the long run value of the target interest rate, x^*, (modified by a negligible convexity term) when maturity (and hence, the number of policy meetings, $n(t,T)$) tends to infinity. The third term is zero for $n(t,T)$ = 0 and also tends to zero when $n(t,T)$ becomes very large. For intermediate maturities, the expression in front of the target interest rate gap, ($x(t)$-x^*), is positive and hence, its impact on the forward rate curve is hump-shaped.

The feature of having a factor with a hump-shaped impact on the forward rate curve (i.e. a curvature factor) is recognized from the Nelson-Siegel factor model. Indeed, if we consider the special case $\gamma = 1$ (see the appendix for derivation) and rearrange we obtain

$$f(t,T) = x^* - c_n^f + (r(t_0) - x^*)\, e^{-\kappa s n(t,T)} + (x(t) - x^*)(1-\omega)\, e^{-\kappa(l(t)-s)}\, n(t,T)\, e^{-\kappa s n(t,T)}, \qquad (23)$$

where we have used the fact that ω equals $e^{-\kappa s}$ in the case γ = 1. Let us compare this expression with the corresponding expression for the Nelson Siegel factor model

$$f(t,T) = \beta_1 + \beta_2(t) e^{-\lambda(T-t)} + \beta_3(t)(T-t)\lambda e^{-\lambda(T-t)}. \qquad (24)$$

We see that expressions (23) and (24) exhibit almost identical functional forms, where $\beta_1, \beta_2(t), \beta_3(t)$, and λ correspond to x^*, $(r(t_0) - x^*)$, $(x(t) - x^*)(1-\omega)\, e^{-\kappa(l(t)-s)}/(\kappa s)$, and κ, respectively. Apart from a negligible convexity term, the difference between (23) and (24) is that time to maturity (T-t) in (24) corresponds to the time length between policy meetings (s) times the number of policy meetings during $[t,T]$ in (23), which is not exactly the same thing, even though $sn(t,T)$ may serve as a reasonable approximation for (T-t), when $n(t,T)$ is large. The observation that it is possible to relate the forward rate curve (23) to the Nelson-Siegel model has been used to obtain rough estimates of parameters β, κ and σ in table 1.[17]

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[17] A comparison between (23) and (24) suggests that $e^{-\lambda(T-t)} = \omega^{n(t,T)} = e^{-\kappa s n(t,T)}$. If we use the λ estimate of 12*0.077 from Diebold, Rudebusch, and Boragan Aruoba (2006) (who measure time in months) together with the values from table 1 for d and s and the observation that $sn(t,T) \approx (T\text{-}t)$ when n is large, the above relationships imply the following estimates: β = 43.6 κ = 0.924. Somewhat different values were used in table 1 in order to avoid the special case γ =1 in examples. Concerning estimates of σ, a comparison between (23) and (24) also suggests that the standard deviation of the β_3 factor takes the form $\sigma_{\beta_3} = \sigma_x (1-\omega)\, e^{-\kappa(l(t)-s)}/(\kappa s)$, where $\sigma_x = \sigma/(2\kappa)$ is the unconditional standard deviation of the target interest rate x. Table 1 in Diebold, Rudebusch, and Boragan

The Nelson-Siegel factor model has been quite popular (see e.g. Diebold and Li (2006) and Diebold, Rudebusch, and Boragan Aruoba (2006)) thanks to the intuitive decomposition of the forward (and spot) interest curve into a level (or asymptotic) factor, β_1, a slope factor, $\beta_2(t)$, and a curvature factor, $\beta_3(t)$. At the same time, the Nelson-Siegel model and, in particular, the curvature term appear to be ad hoc. However, the above analysis suggests that it seems possible to derive a modified Nelson-Siegel factor model that is based on modern asset pricing theory when the policy rate process is given by (4).

Gradualism

The quantity ω = 1-βd plays an important role in the above analysis as seen from expression (19). There is another way of assessing the magnitude of ω and, at the same time, present a somewhat new view of the role of gradualism and interest rate smoothing. Assume that we use quarterly data in order to estimate a policy rule of the following type:

$$r_t = (1\text{-}\rho)(r^* + \eta g_t) + \rho r_{t-1} + \varepsilon_t, 0 \leq \rho < 1, \tag{25}$$

where r_t denotes quarterly observations of the actual policy process $r(t)$, g_t is a vector of macroeconomic gaps to which monetary policy responds and where, r^*, η (vector), and ρ are parameters. How does monetary policy rule according to (4) relate to policy rule (25)? First, r^* is the long run value of r_t and we see from (4) that the short term interest rate is expected to converge towards x^*, so $r^* = x^*$. Moreover, parameter η determines how sensitive r_t is to economic developments g_t and the term ηg_t corresponds to the target interest gap, $x(t)-x^*$. However, we are mostly interested in the last term, ρr_{t-1}, where parameter ρ is often interpreted as a measure of interest rate smoothing. Moreover, we see from (19) that the term $\omega^{n(t,T)} r(t_0)$ determines the impact on the (expected) future policy rate of the current policy rate. In other words,

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Aruoba (2006) indicates that $\beta_3(t)$ (denoted by C_t) may be approximated by an AR(1)-process, in which case we obtain the estimate σ_{β_3} = 0.024 (if interest rates are measured in decimal form). This estimate together with the other parameter values render an estimate σ = 0.033.

$$\rho \approx w^{n(t,T)}, \tag{26}$$

where the number of policy meetings between quarterly observations of the short term interest rate, $n(t,T)$, should be two on average (when $s = 1/8$ years). Relation (26) enables us to obtain rough estimates of the crucial quantity ω. According to Clarida, Gali and Gertler (2000), a typical estimate of the smoothing parameter ρ seems to be around 0.8, which corresponds to a ω value of 0.89 ($n(t,T)$ = 2). This is close to the estimate of ω that was obtained when the forward rate curve (23) was compared to the Nelson-Siegel curve (24); see footnote 17. To sum up: The size of the important quantity $w = 1 - \beta d$ can be assessed by analysing parametric models of the forward rate curve as well as by time series analysis of policy rules, and the fact that these two very different approaches give similar estimates of w is reassuring.

Next, we turn to the effects of monetary policy shocks defined as unexpected changes in the policy rate. Formally, such a shock at policy date t_k can be written as $\Delta r(t_k) - E_{t_{k-}}[\Delta r(t_k)]$, where $\Delta r(t_k)$ and $E_{t_{k-}}[\Delta r(t_k)]$ are the actual and the expected (just before the policy meeting) adjustment of the policy rate, respectively. To analyse term structure effects of such shocks, let us define $\Delta f(t_k, T)$ as the shift in forward rates associated with a policy decision taken at time t_k. It is straightforward to show that this shift is given by

$$\Delta f(t,T) = \omega^{n(t_{k-},T)-1} \left(\Delta r(t_k) - E_{t_{k-}}[\Delta r(t_k)] \right), \tag{27}$$

where one realizes that the number of policy meetings during $[t_{k-}, T], n(t_{k-}, T)$, includes the meeting at t_k. Moreover, from (4), we also have that

$$E_{t_{k-}}[\Delta r(t_k)] = \beta d z(t_{k-}) . \tag{28}$$

An interesting example of a policy shock is when there are expectations about an adjustment in the policy rate (say upwards), but the decision is to leave the policy rate unchanged. For instance, assume that the (pre-meeting) policy gap is two percentage points which, according to (28) (using parameter values from table 1), implies that the expected adjustment of the policy rate is 20 basis points. In this case, market participants expect an upward adjustment of the policy rate, but a decision not to change the policy rate is also a possibility. Further, assume that two policy meetings will be held during the three coming months with the next policy meeting in a very near future. A decision not to change the policy rate will then decrease a three-month forward interest rate by minus 18 basis points according to (27). Notice that this effect arises despite the fact that the state variables (x and r) do not change! Obviously, many affine models will have a hard time explaining such dynamics since factors causing term structure movements are normally unchanged in this example.[18] The shift in the forward rates is only due to the fact that the number of policy meetings during maturity ($n(t,T)$) has decreased by one (and the time to the next policy meeting has increased from 0 to s years). This example clearly illustrates the importance of calendar effects.

There are some caveats to this example. First, the fact that the target interest rate is treated as an exogenous variable is a simplification, even though the quantitative effects of relaxing this assumption are probably small. Moreover, the unexpected decision to leave the policy rate unaffected is often associated with new intentions from the central bank which, in turn, also affect monetary policy expectations and the term structure of interest rates. We will shed some light on these issues later.

Before we consider some extensions of the above term structure model, let us take a small look at zero coupon yields. Zero coupon yields, $y(t,T)$, can easily be computed from (9) as $y(t,T) = -\ln(B_n(t,T))/(T-t)$, i.e.

$$y(t,T) = [\,\varphi_n(l(t),\tau)\; r(t_0) + \eta_n(\tau)\, x^* + \pi_n(l(t),\tau)\,(x(t)-x^*) - c_n(l(t),\tau)\,]/(T-t), \qquad (29)$$

where $\varphi_n(l(t),\tau)$, $\eta_n(\tau)$, $\pi_n(l(t),\tau)$, and $c_n(l(t),\tau)$ are given by expressions (11)-(17). No additional information or genuine theoretical insights are gained from this expression, but it plays a potentially important role in empirical work since price information from the term structure is very often quoted in terms of yields. Moreover, the forward rate expression (19) is essentially only a function of the number of policy meetings, $n(t,T)$, during $[t,T]$, whereas a closer inspection of (29) reveals that we also need information about time after the last

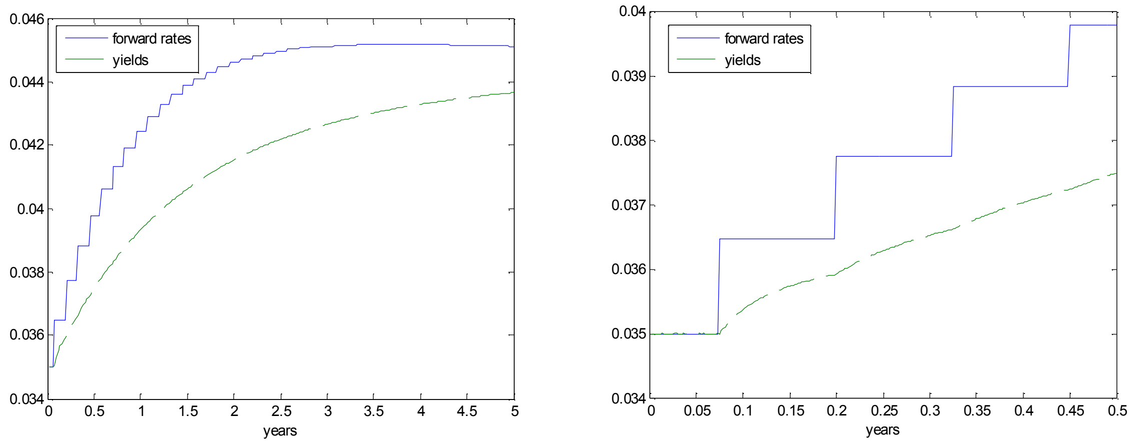
[18] Naturally, those affine models, the factors of which are pure term structure factors, e.g. the level, slope and curvature model of Cochrane and Piazzesi (2005), will be able to at least partially account for those effects.

meeting (τ) to calculate yields. To illustrate this point, let us assume that the target interest rate, x, equals its long run value, x^*, but that the current policy rate is two percentage points below the target interest rate at 2.50 percent. Let us consider a one- month interest rate where one policy meeting will take place just before the maturity date of the underlying bond. This means that the time after the last meeting (τ) is essentially zero which, in turn, implies that the one-month yield more or less coincides with the current policy rate of 2.50 percent. Now we move (almost) one month forward to just before the coming policy meeting and assume that no additional policy meeting will occur until maturity and that the target interest rate remains at its long run value. In this case, the time after the last meeting (τ) is essentially the same as the time to maturity (T-t), which together with the parameter values in table 1 imply a one-month yield of 2.70 percentage points. Despite the fact that nothing has happened besides the flow of time, the one-month yield has increased by 20 basis points! Once again, we have an example where calendar time effects are substantial.

We also demonstrate graphically what the forward rate and the yield curve can look like according to the term structure model, see figure 1. The graphs are based on the assumption that the current policy rate is 3.5 percent, which is below the current target interest rate of 5 percent so there is a positive policy gap and hence, a tendency to a hump-shaped forward rate curve, see figure 1a. The step-shaped pattern is evident, especially at the short end of the forward rate curve, but otherwise the curves look very normal. However, if we shrink the time horizon to six months, the step-shaped forward rate curve becomes a dominant feature that cannot be reproduced by traditional term structure models. Moreover, the yield curve exhibits some non-standard features. The very short end is flat and a (weak) local convex shape between the first policy meetings can be observed.

Figure 1. The forward rate curve and the yield curve

a) Time horizon: 5 years **b) Time horizon: 6 months**



Note. The construction of the graphs is based on the parameter values in table 1 and the following initial conditions: $r = 0.035$, $x = 0.05$, and $l(t) = 0.075$ years (time to next policy meeting).

Extension to the multifactor target interest rate case

The assumption that the target interest rate, x, is driven by one factor was made for simplicity and it is straightforward to extend the model to the case when the target interest rate is driven by multiple factors. More precisely, let us assume that the target interest rate, x, is the sum of two components according to

$$x = x_1 + x_2 , \tag{30}$$

$$\begin{pmatrix} dx_1(t) \\ dx_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ \kappa_2 x^* \end{pmatrix} - K \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_1 dW_1(t) \\ \sigma_2 dW_2(t) \end{pmatrix} \tag{31}$$

$$K = \begin{pmatrix} \kappa_1 & 0 \\ \rho & \kappa_2 \end{pmatrix}, \tag{32}$$

where $dW_1(t)$ and $dW_2(t)$ are independent increments of standard Wiener processes. The state variable x_2 is the "normal" target interest rate, i.e. the interest rate that the central bank adjusts towards normally. It has the same long run value, x^*, as the target interest rate process described by (1). However, the target interest rate is influenced by monetary shocks (x_1) called intention shocks, i.e. the central bank signals that for a while, it aims at a target interest rate (x) that differs from the normal target interest rate (x_2). These kinds of intentional shocks can, in principle, occur anytime, e.g. in the form of signalled intentions in speeches. In the appendix, it is shown that the expected future target interest rate is given by

$$E_t[x(t+u)] = (1+\varphi)\ x_1(t)\ e^{-\kappa_1 u} + [\, x_2(t) - x^* - \varphi x_1(t)\,]\, e^{-\kappa_2 u}, \quad \varphi = \frac{\rho}{\kappa_1 - \kappa_2}, \tag{33}$$

provided that $\kappa_1 \neq \kappa_2$. Expression (19) in this case extends to

$$f(t,T) = \omega^{n(t,T)} r(t_0) + (1\text{-}\omega)^{n(t,T)} x^* + \theta_1(l(t))\,(e^{-\kappa_1 s n(t,T)} - \omega^{n(t,T)})\ (1+\varphi)\ x_1(t)$$
$$+\theta_2(l(t))\,(e^{-\kappa_2 s n(t,T)} - \omega^{n(t,T)})\,[\, x_2(t) - x^* - \varphi x_1(t)\,] - c_n^f, \tag{34}$$

where $\theta_i(l(t)) = (1\text{-}\omega)\, e^{-\kappa_i (l(t)-s)} / (1\text{-}\gamma_i)$, $\gamma_i = \omega e^{\kappa_i s}$, $i = 1,2$, and c_n^f is a small convexity term. It is easily verified that the case $\gamma_1 = \gamma_2 = 1$ corresponds to the extended Nelson-Siegel model (or the Svensson model, see Svensson (1995)). The normal target interest rate should to a large extent reflect macroeconomic conditions that, in turn, are affected by intention shocks. If parameter ρ is positive, then an intention shock will have a dampening effect on the normal target interest rate (x_2). In this sense, the target interest rate is no longer treated as an exogenous state variable.

Another possibility is to let the target interest rate be a continuous time Taylor rule, i.e. $x = x^* + g_\pi(\pi\text{-}\pi^*) + g_y(y\text{-}y^p)$, where $(\pi\text{-}\pi^*)$ and $(y - y^p)$ denote the inflation gap (relative to an inflation target π^*), and the output gap (deviation from potential output y^p), respectively. Naturally, the factor dynamics (for state variables $(\pi\text{-}\pi^*)$ and $(y - y^p)$) must be modified, in particular matrix K in (31).

Overnight spreads and the implementation lags

In reality, the policy rate does not necessarily exactly represent the instantaneous risk-free rate. Market participants may face an overnight rate, $r^O(t)$, that differs slightly from the policy rate. However, it is straightforward to adjust for this discrepancy. Assume that the spread, δ, between the overnight rate and the policy rate follows an Ornstein-Uhlenbeck process according to

$$d\delta(t) = \kappa_\delta\ [\ \delta(t)\text{-}\delta^*]\, dt + \sigma_\delta\ dW_\delta(t)\,, \tag{35}$$

where $dW_\delta(t)$ is an increment of a standard Wiener process independent of anything else. If we replace the policy rate, $r(t)$, by the overnight rate, $r^o(t) = r(t) + \delta(t)$, in the valuation formula (8), we obtain a modified bond price

$$B(t,T) = B_0(t,T)\, e^{-\delta^*(T-t)-(\delta(t)-\delta^*)\xi(T-t,\kappa_\delta)+0.5^*\Omega(T-t,\kappa_\delta,\sigma_\delta)}\,, \tag{36}$$

where functions ξ and Ω are defined as

$$\xi(u,\kappa) = (1\text{-}e^{-\kappa u})/\kappa,\ \ \Omega(u,\kappa,\sigma) = \sigma^2[u-2\xi(u,\kappa)+\xi(u,2\kappa)]/\kappa^2\,, \tag{37}$$

where $B_0(t,T)$ is the bond price when there is no difference between the overnight rate and the policy rate. It easily verified (and quite obvious) that the modified expression for the yield is the previous one (see (29)), plus a small Vasicek component.[19] The additional spread component δ can be quite volatile and adds substantial volatility to the short end of the yield curve.

Next, we consider the case when there is an implementation lag (ε), i.e. a policy rate decision announced at policy meeting date t_k is implemented at date $t_k^\varepsilon = t_k + \varepsilon$, where ε is a small positive number. Let $B_n^\varepsilon(t,T)$ be the price of a zero coupon bond at time t, $t_0^\varepsilon \le t < t_1^\varepsilon$, when there are n policy rate *implementations* during $[t,T]$ and an implementation lag of ε. In the normal case when $t_0^\varepsilon \le t < t_1$, it can be shown that the bond price, $B_n^\varepsilon(t,T)$, takes the form[20]

$$B_n^\varepsilon(t,T) = e^{-r(t_0^\varepsilon)\varepsilon}\, B_n(t,T-\varepsilon) = e^{-r(t_0^\varepsilon)\varepsilon}\, \overline{B}_n(l(t),T-\varepsilon,r(t_0^\varepsilon))\,, \tag{38}$$

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[19] See equation (29) in Vasicek (1977). Notice that we have set the market price for the overnight rate risk at zero, but an extension to the case when the market price for overnight risk is no-zero is straightforward; see footnote 13.

[20] To understand (38), an additional deterministic factor $\exp(-r(t_0^\varepsilon)\varepsilon)$ can be factored out from the stochastic discount expression appearing within brackets in (8), due to the implementation lag. The remaining stochastic part of the discount factor then has the same form as before (i.e. the case with no implementation lag) with the exception that the last part of the step-wise integral in (8), i.e. the part representing the impact of the nth and last policy rate decision, is smaller since the time after the last implementation (τ) is ε years shorter.

where B_n refers to the corresponding bond pricing formula when there is no implementation lag, e.g. formula (9). Function $\overline{B}_n$ represents the same valuation formula, but it highlights how the bond price depends on the time to the next policy meeting, $l(t)$, and the current policy rate, $r(t_0^{\varepsilon})$. The reason for this more ambitious notation will become clear shortly. The modified bond pricing formula is quite intuitive. The implementation lag means that the current policy rate will prevail somewhat longer and its impact will be somewhat larger on the bond price, as reflected in the exponential factor, whereas the impact of the expected policy path after the next policy rate decision is somewhat smaller as reflected in the slight reduction of the maturity in the bond price function $\overline{B}_n$.

The bond price formula become slightly more complicated when $t_1 \le t < t_1^{\varepsilon}$, i.e. we know the policy rate decision announced at time t_1 but it has not yet been implemented. The modified bond price in this case, $\hat{B}_n^{\varepsilon}(t,T)$, can be written as

$$\hat{B}_n^{\varepsilon}(t,T) = e^{\Delta r(t_1^{\varepsilon})(t_1^{\varepsilon}-t)}\ B_n^{\varepsilon}(t,T) = e^{\Delta r(t_1^{\varepsilon})(t_1^{\varepsilon}-t)}\ e^{-r(t_1^{\varepsilon})\varepsilon}\ \overline{B}_n(t_2 - t, T-\varepsilon, r(t_1^{\varepsilon})), \tag{39}$$

where $\Delta r(t_1^{\varepsilon}) = r(t_1^{\varepsilon}) - r(t_0^{\varepsilon})$. To understand (39), notice that in the case $r(t_1^{\varepsilon}) = r(t_0^{\varepsilon})$, we have returned to the previous case with the minor modification that all policy dates have been updated as underscored by the arguments in function $\overline{B}_n$. If $r(t_1^{\varepsilon}) \ne r(t_0^{\varepsilon})$, we must adjust the formula by the deterministic factor $\exp(\Delta r(t_1^{\varepsilon})(t_1^{\varepsilon}-t))$.

The correction for the implementation lag is mainly of importance for short term interest rates for which $n(t,T) = 1$. In this case, the impact on a short yield of the implementation lag can be approximated by $-\varepsilon E_t[\Delta r(t_1^{\varepsilon})]/(T-t)$. For instance, ECB operates with an implementation lag of one week and if the expected policy rate change is 25 basis points, the implementation lag causes a downward adjustment of about 6 basis points on a one-month yield.

4. Conclusions.

This paper has derived and presented a model of the term structure of interest rates, where the policy rate of the central bank evolves in accordance with the monetary policy decision-making process, i.e. the policy rate is only adjusted upward or downward at pre-announced dates by a small fixed amount (or a multiple thereof). The most noticeable result is that spot and forward rates become explicit functions of the number of policy meetings during the life of the underlying bond.[21] As a consequence, the forward rate curve is step-shaped, which reflects the fact that the expectations of the future policy rate are constant between policy meetings. Moreover, for a given time to maturity, the number of policy meetings until maturity will depend on the position in the policy cycle. In other words, there are calendar time effects that can have a substantial impact on short term spot and forward rates. For instance, the spot and forward rate curve may shift at policy meetings even if the state variables (including the policy rate) are unchanged since the number of policy meetings will drop by one for all maturities. Furthermore, given the number of policy meetings until maturity, the short end of the yield curve will heavily depend on when the next policy meeting takes place. The calendar time effects discussed above are obvious for central bank watchers, but they are nevertheless not included in the vast majority of term structure models.

The presented model also has important implications for longer spot and forward rates. In particular, it turns out that economic shocks will have a hump-shaped impact on the forward rate curve (and hence, also on the yield curve); a feature that is in accordance with the empirical findings. A somewhat surprising finding is that a modified version of the Nelson-Siegel model of the forward rate curve is obtained as a special case. Another observation is that the dynamic process for the policy rate assumed in the model also seems to explain why the lagged policy rate appears in estimated policy rules. It is also relatively straightforward to extend the model by including more state variables, e.g. macroeconomic variables.

For future research, an extension of the model where macroeconomic variables enter endogenously and more explicitly would be interesting. Another important (but difficult) issue is to examine the term structure implications of assuming a non-linear relationship

[21] Naturally, the number of policy meetings is a function of time to maturity, so spot and forward rates are implicit functions of time to maturity.

between the policy gap and the expected change in the policy rate, e.g. a model where adjustment probabilities are given by a probit model.[22] The pricing of interest rate derivatives under the short interest rate dynamics assumed in this paper is another natural field of research. Finally, estimating the term structure model using time series of term structure data and thereby shedding some additional light on the empirical relevance of the model and calendar time effects in particular is, of course, a very interesting research project.

[22] See e.g. Gerlach (2005), who analyses the ECB policy rate process empirically by an ordered probit model.

Appendix1. Derivation of the expected future policy rate.
All we need to do is to derive expressions for $E_t[r(t_n)]$ for all positive integers n. Direct calculations give that

$$E_t[r(t_1)] = r(t_0) + (1\text{-}\omega) g(t_1), \tag{A1}$$

where $\omega = 1\text{-}\beta d$ and function g has the form

$$g(u) = e^{-\kappa(u-t)} (x(t)\text{-} x^*) + x^*\text{-} r(t_0). \tag{A2}$$

Next, we have that

$$E_t[z(t_{2-})] = g(t_2) - (1\text{-}\omega) g(t_1), \tag{A3}$$

where $z(t_{2-})$ is defined by (7). Thus,

$$E_t[r(t_2)] = E_t[r(t_1)] + (1\text{-}\omega)\, E_t[z(t_2)] = r(t_0) + (1\text{-}\omega)\, \omega g(t_1) + (1\text{-}\omega) g(t_2). \tag{A4}$$

By induction, it is straightforward to show that expressions (A1) and (A4) can be generalized to

$$E_t[r(t_n)] = r(t_0) + (1\text{-}\omega) \sum_{k=0}^{n-1} \omega^k g(t_{n-k}). \tag{A5}$$

Substituting the definition of function g into (A5) and rearranging gives

$$E_t[r(t_n)] = r(t_0) + (1\text{-}\omega) \sum_{k=0}^{n-1} [\mu_n(t)\gamma^k (x(t) - x^*) + \omega^k (x^* - r(t_0))], \tag{A6}$$

where $\mu_n(t) = e^{-\kappa(l(t)+(n-1)s)}$ and $\gamma = \omega e^{\kappa s}$. Using the formula for geometric series yields

$$E_t[r(t_n)] = \omega^n r(t_0) + (1\text{-}\omega^n) x^* + \theta(l(t))(e^{-\kappa s n} - \omega^n)\, (x(t)\text{-}x^*). \tag{A7}$$

In the case $\gamma = 1$, (A6) can be written as

$$E_t[r(t_n)] = r(t_0) + (1\text{-}\omega)\, \mu_n(t)\, n(x(t) - x^*) + (1 - \omega^n)(x^* - r(t_0)). \tag{A8}$$

Using the fact that $\gamma = 1$ implies $\omega = e^{-\kappa s}$, (A7) can easily be rearranged to

$$E_t[r(t_n)] = x^* + (r(t_0)\text{-}x^*)\, e^{-\kappa s n} + (x(t)\text{-}x^*)\, (1\text{-}\omega)\, e^{-\kappa(l(t)-s)}\, n e^{-\kappa s n}. \tag{A9}$$

Appendix 2. The integral of the policy rate path
Given expressions for the expected future policy rate, we then calculate the expected integral of the future policy rate for different time horizons, i.e. we define

$$J_0(t,T) = r(t_0)\,(T\text{-}t) \tag{A10}$$

$$J_1(t,T) = r(t_0)\,(t_1\text{-}t) + E_t[r(t_1)]\,(T\text{-}t_1) \tag{A11}$$

$$J_n(t,T) = r(t_0)\,(t_1\text{-}t) + s\sum_{k=1}^{n-1} E_t\big[r(t_k)\big] + E_t[r(t_n)]\,(T\text{-}t_n),\ n = 2,3,\ldots \tag{A12}$$

For n = 1,2,3 …, we can use (A6) and the formula for geometric series and obtain (10).

Appendix 3. Derivation of the bond pricing formula (9)

We demonstrate the validity of formula (9) by induction. Given that $r(t_0) = r$ is observed, we trivially have that

$$B_0(t,T) = e^{-r(T-t)}, \tag{A13}$$

which is in accordance with formula (9). Next, we notice that

$$B_1(t,T) = e^{-r(t_1-t)}\, E_t[e^{-r(t_1)(T-t_1)}] = e^{-r(T-t)}\, E_t\left[e^{-d(T-t_1)(N^+_{t_1}-N^-_{t_1})}\right]. \tag{A14}$$

We calculate (A12) by first defining $B_1^z(t,T) = e^{-r(T-t)}\, E_t\left[e^{-d(T-t_1)(N^+_{t_1}-N^-_{t_1})} | z(t_{1-}) = z\right]$.

Moreover, $B_1^z(t,T)$ can be written as

$$B_1^z(t,T) = e^{-r(T-t)}\, B_1^+(t_1,T)\, B_1^-(t_1,T), \tag{A15}$$

where

$$B_1^+(t_1,T) = E_t\left[e^{-d(T-t_1)N^+_{t_1}} | z(t_{1-}) = z\right],\ B_1^-(t_1,T) = E_t\left[e^{d(T-t_1)N^-_{t_1}} | z(t_{1-}) = z\right]. \tag{A16}$$

Now

$$B_1^+(t_1,T) = \sum_{k=0}^{\infty} e^{-\lambda^+(z)} \frac{\lambda^+(z)^k\, e^{-kd(T-t_1)}}{k!} = e^{\lambda^+(z)(e^{-d(T-t_1)}-1)}. \tag{A17}$$

Similarly, we have that

$$B_1^-(t_1,T) = e^{\lambda^-(z)(e^{d(T-t_1)}-1)}. \tag{A18}$$

Using (6) and the approximation $e^u - 1 \approx u$, it is straightforward to verify that (A15), (A17) and (A18) together imply that

$$B_1^z(t,T) = e^{-r(T-t)+h(T-t_1)-(1-\omega)(T-t_1)z}, \tag{A19}$$

where

$$h(u) = 2\lambda_0\,[\cosh(du) - 1], \qquad \cosh(u) = 0.5[e^u + e^{-u}]. \tag{A20}$$

To calculate the bond price $B_1(t,T) = E_t[B_1^z(t,T)]$, we use fact that $z(t_{1-}) \sim N(m(t_1-t)-r,\ v^2(t_1-t))$, where functions m and v^2 are given by (2), to get

$$B_1(t,T) = e^{-r(T-t)+h(T-t_1)-(1-w)\tau(m(t_1-t)-r)+(1-w)^2\tau^2 v^2(t_1-t)/2}. \tag{A21}$$

A closer inspection of (A19) confirms that this is in accordance with formula (9) for $n = 1$. Finally, we need to validate the formula for n+1, provided that it is true for 1,2,…,n. First, we have

$$B_{n+1}(t,T) = e^{-r(t_1-t)}\; E_t\big[B_n(t_1,T)\big] = e^{-r(t_1-t)+c_n(s,T)}\; E_t[e^{-J_n(t_1,T)}]. \tag{A22}$$

When analysing $J_n(t_1,T)$ in (A22), it is important to realize that the time to the next meeting and the time after last meeting take the form $l(t_1) = t_2 - t_1 = s$ and $\tau = T - t_{n+1}$, respectively. Thus,

$$J_n(t_1,T) = \varphi_n(s,\tau)\,(r + d[N^+_{t_1} - N^-_{t_1}]) + \eta_n(\tau)\,x^* + \pi_n(s,\tau)\,(x(t_1) - x^*), \tag{A23}$$

where

$$\varphi_n(s,\tau) = s\frac{(1-\omega^n)}{1-\omega} + \tau\omega^n, \tag{A24}$$

$$\eta_n(\tau) = s[(n\text{-}1) - \frac{\omega(1-\omega^{n-1})}{1-\omega}] + \tau\,(1\text{-}\omega^n), \tag{A25}$$

$$\pi_n(s,\tau) = \theta(s)\,[s\omega\left(\frac{1-(\omega/\gamma)^{n-1}}{\gamma-\omega} - \frac{(1-\omega^{n-1})}{1-\omega}\right) + \tau\,((\omega/\gamma)^n - \omega^n)], \tag{A26}$$

$$\theta(s) = \frac{(1-\omega)}{(1-\gamma)}. \tag{A27}$$

As before, we first fix $x(t_1) = x$ (and hence also $z(t_1) = z = x\text{-}r$) and define $B^z_n(t_1,T) = E_t[e^{-J_n(t_1,T)} | z(t_{1-}) = z]$, which can be written as

$$B^z_n(t_1,T) = e^{-\varphi_n(s,\tau)r - \eta_n(\tau)x^* - \pi_n(s,\tau)(x-x^*)} B^+_n B^-_n, \tag{A28}$$

where

$$B^+_n = E_t\left[e^{-d\varphi_n(s,\tau)N^+_{t_1}}\right] = e^{\lambda^+(z)(e^{-d\varphi_n(s,\tau)}-1)} \tag{A29}$$

and

$$B^-_n = E_t\left[e^{d\varphi_n(s,\tau)N^-_{t_1}}\right] = e^{\lambda^-(z)(e^{d\varphi_n(s,\tau)}-1)}. \tag{A30}$$

Using the approximation $e^u - 1 \approx u$, we obtain

$$B^+_n\; B^-_n = e^{h(d\varphi_n(s,\tau)) - \beta d\varphi_n(s,\tau)z}, \tag{A31}$$

where the function h is given by (A20). (A31) substituted into (A28) yields

$$B^z_n(t_1,T) = e^{-\omega\varphi_n(s,\tau)r - \eta_n(\tau)x^* - \pi_n(s,\tau)(x-x^*) - (1-w)\varphi_n(s,\tau)x + h(d\varphi_n(s,\tau))}. \tag{A32}$$

Now we treat $x(t_1)$ as a random variable normally distributed according to (2) and obtain

$$E_t\left[e^{-J_n(t_1,T)}\right] = E_t\left[B^z_n(t_1,T)\right]$$
$$= e^{-\omega\varphi_n(t_1)r - \eta_n(t_1)x^* - \pi_n(t_1)e^{-\kappa(t_1-t)}(x(t)-x^*) - (1-w)\varphi_n(t_1)(x^* + e^{-\kappa(t_1-t)}(x(t)-x^*)) + \Lambda_n} \tag{A33}$$

$$\Lambda_n = h(d\varphi_n(s,\tau)) + ((1-\omega)\varphi_n(s,\tau) + \pi_n(s,\tau))^2 \nu^2 (t_1 - t)/2. \tag{A34}$$

It is a straightforward exercise to verify that (A33) and (A34) substituted into (A22) lead to expressions (9), when the number of policy meetings during $[t,T]$ is n+1, which was to be shown.

Appendix 4. Assessment of the approximation error for yields

We consider a bond at time t that will experience n policy meetings before it matures at time T and define $|\Delta i_n(t,T)|$ as the absolute value of the impact on the yield expression to which the approximation gives rise. Since the approximation only affects the factor $B_n^z(t_1,T)$, one realizes that (for a fixed and positive value of the policy gap z)

$$|\Delta i_{n+1}(t,T)| = \left|\ln(\hat{B}_n^z(t_1,T)/B_n^z(t_1,T)\right| / (T\text{-}t) < n|\beta z g(s/\beta)|/(T\text{-}t)$$
$$< |zg(s/\beta)|\,\beta/s = |z|\,k(s/\beta),\ n=1,2,3,\ldots, \qquad \text{(A35)}$$

where a hat indicates that the exact expression for the bond factor, $B_n^z(t_1,T)$, is used and functions g and k are given by

$$g(u) = u + e^{-u} - 1,\quad k(u) = g(u)/u,\ u > 0 \qquad \text{(A36)}$$

when $z > 0$. The first inequality is based on the observation that the expression $\left|\ln(\hat{B}_n^z/B_n^z)\right|$ involves n approximations of type $e^u - 1 \approx u$, and all these approximations are bounded from above by function g evaluated at s/β.[23]. The last inequality follows from the fact that $(T\text{-}t) > ns$. It is easily verified that k is an increasing function. If s = 1/8 and β = 40 from table 1 is used in (A35), we obtain that $|\Delta i_{n+1}(t,T)| < 0.0016|z|$. From this example, it is clear that it is very unlikely to get an upper bound of the approximation error exceeding 1 basis point, even if less favourable values of s and β are used. Normally, the error should be less than 0.5 basis points and therefore, not noticeable.

Appendix 5. Derivation of expression (33).

A standard result, see e.g. section 5.6 in Karatzas and Shreve (1988), is that the expected value of the state vector $w = (x_1, x_2)$, in (31) is given by

$$E_t[w(t+u)] = \Phi(u)w(t) + \Phi(u)\int_t^{t+u}\Phi(-v)b\,dv, \qquad \text{(A37)}$$

where $b = (0, \kappa_2 x^*)'$, and the 2x2 matrix Φ fulfils

$$\dot{\Phi} = -K\Phi,\ \Phi(0) = I, \qquad \text{(A38)}$$

where I is the identity matrix and a dot denotes a derivative with respect to time. It is a straightforward exercise to show that the solution to (A38) is given by

$$\Phi(u) = \begin{pmatrix} e^{-\kappa_1 u} & 0 \\ \varphi(e^{-\kappa_1 u} - e^{-\kappa_2 u}) & e^{-\kappa_2 u}\end{pmatrix},\quad \varphi = \rho/(\kappa_1 - \kappa_2). \qquad \text{(A39)}$$

Inserting (A39) into (A37) and multiplying by the vector (1,1)' from the left gives (33).

[23] A comparison between (A31) and the exact expression reveals that a term $\exp(g(d\varphi_n(s,\tau)))$ is ignored. Moreover, it is easily seen that $\varphi_n(s,\tau)$ is bounded above from $s/(1\text{-}w) = s/(\beta d)$. Since g is an increasing function, it follows that $\exp(g(d\varphi_n(s,\tau))) < \exp(g(s/\beta))$.

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