Technical Appendix to "The New Keynesian Phillips Curve and Staggered Price and Wage Determination in a Model with Firm-Specific Labor"*

Mikael Carlsson[†] and Andreas Westermark[‡]

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Abstract

In this paper we describe in detail how to derive the model used in the paper "The New Keynesian Phillips Curve and Staggered Price and Wage Determination in a Model with Firm-Specific Labor". We present the model, log-linearize it, derive a model-consistent welfare criterion and proceed to solve the model, both if policy follows a simple rule and if monetary policy is optimal.

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[†]Research Department, Sveriges Riksbank, SE-103 37, Stockholm, Sweden. e-mail: mikael.carlsson@riksbank.se.

[‡]Research Department, Sveriges Riksbank, SE-103 37, Stockholm, Sweden. e-mail: andreas.westermark@riksbank.se.

1 Introduction

In this paper the model in "The New Keynesian Phillips Curve and Staggered Price and Wage Determination in a Model with Firm-Specific Labor" is presented in detail. We describe the agents and sectors of the economy and state the conditions for optimizing behavior. We then describe how to compute the steady state of the model. Next, we proceed to log-linearize the flexible as well as the sticky price model around the steady state. We first log-linearize the optimal price- and wage-setting decisions. We then proceed to derive a second-order log approximation of the welfare function of the sticky price model. Finally, we solve the model, both if policy follows a simple rule and if monetary policy is optimal.

In section 2, we outline the model. In section 3 and 4 we log-linearize the flexible and sticky price models, respectively, in section 5 the log quadratic approximation of welfare is derived and in 6 the model is solved. Finally, in section 7 the Erceg, Henderson & Levin (2000) model is described and solved.

2 The Economic Environment

There is a competitive final goods sector with flexible prices and a monopolistically competitive intermediate goods sector where producers set prices in staggered contracts as in Calvo (1983). Intermediate goods producers set prices in staggered contracts as in Calvo (1983). In order to introduce complete consumption insurance we rely on a representative family as in Merz (1995), that consists of a large number of households. To each firm a household is attached. Thus, in contrast to Erceg et al. (2000), firms do not perceive workers as atomistic. In each period, wages are renegotiated with a fixed probability. Thus, wages are staggered as in Calvo (1983) but, in contrast to Erceg et al. (2000), they are determined in bargaining between a union and the firm and not unilaterally by the union.

2.1 Final goods firms

Since we assume complete insurance, using a representative family as in Merz (1995), households are identical, except for leisure choices. It then simplifies the analysis to abstract away from the households optimal choices for individual goods. We follow Erceg et al. (2000) and assume a competitive sector selling a composite final good. The composite good is combined from individual or intermediate goods in the same proportions that households would choose. The composite good is

$$Y_t = \left[\int_0^1 Y_t\left(f\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\sigma > 1$ and $Y_t(f)$ is the intermediate good produced by intermediate goods firm f. The price P_t of one unit of the composite good is set equal to marginal cost

$$P_t = \left[\int_0^1 P_t(f)^{1-\sigma} df \right]^{\frac{1}{1-\sigma}} . \tag{2}$$

2.2 Intermediate good firms

By standard arguments, the demand function for the generic good f from the final goods sector is

$$Y_{t+k}(f) = \left(\frac{\bar{\pi}^k P_t(f)}{P_{t+k}} \right)^{-\sigma} Y_{t+k}. \tag{3}$$

Intermediate goods firms produce according to the following constant returns production function

$$Y_t(f) = A_t L_t(f)^{1-\gamma}, \tag{4}$$

where A_t is the technology level, common to all firms, and $L_t(f)$ denote the firms labor input in period t. Since firms choose employment unilaterally, $L_t(f)$ are chosen optimally, taking the wage $W_t(f)$ as given. Solving for labor choice in the cost minimization problem trivially gives,

$$L_t(f) = \left(\frac{Y_t(f)}{A_t} \right)^{\frac{1}{1-\gamma}} . \tag{5}$$

The cost and marginal cost functions for firm f are then given by

$$\begin{aligned} TC(W_t(f), Y_t(f)) &= W_t(f) \left(\frac{Y_t(f)}{A_t} \right)^{\frac{1}{1-\gamma}}, \\ MC(W_t(f), Y_t(f)) &= \frac{1}{1-\gamma} \frac{W_t(f)}{Y_t(f)} \left(\frac{Y_t(f)}{A_t} \right)^{\frac{1}{1-\gamma}}, \end{aligned} \tag{6}$$

respectively. The marginal product is in real terms, ignoring the time period when the contract was signed,

$$MPL_t(f) = (1-\gamma) A_t L_t(f)^{-\gamma} = (1-\gamma) \frac{Y_t(f)}{L_t(f)}, \tag{7}$$

where $w_t(f) = \frac{W_t(f)}{P_t}$ is the real wage.[1] Furthermore, real costs is given by

$$tc\left(w_t(f), Y_t(f)\right) = \left(\frac{Y_t(f)}{A_t}\right)^{\frac{1}{1-\gamma}} w_t(f). \tag{8}$$

2.3 Calvo price and wage determination with indexation

Firms are allowed to change prices in a given period with probability $1-\alpha$ and to renegotiate wages with probability $1-\alpha_w$. Any firm that renegotiates wages, is also allowed to change prices. The probability that prices are unchanged is $\alpha_w\alpha$. This assumption simplifies our problem greatly, since it eliminates any intertemporal interdependence in price-setting decisions for a given firm. We assume that prices are indexed by the steady-state inflation rate, as in Yun (1996).

2.3.1 Prices

The producers choose prices to maximize

$$\max_{P_t(f)} E_t \sum_{k=0}^{\infty} (\alpha_w\alpha)^k \Psi_{t,t+k} \left[(1+\tau)\bar{\pi}^k P_t(f) Y_{t+k}(f) - TC\left(W_t(f), Y_t(f)\right)\right] \tag{9}$$

$$\text{s. t.} \quad Y_{t+k}(f) = \left(\frac{\bar{\pi}^k P_t(f)}{P_{t+k}}\right)^{-\sigma} Y_{t+k}.$$

Note that the term within the square brackets is just the firm's profit in period $t+k$, given that prices were last reset in period t. The term $\Psi_{t,t+k}$ captures households valuation of nominal profits in period $t+k$. This will in general depend on time preferences β^k and the marginal utility in period $t+k$. The first-order condition is

$$\digamma = E_t \sum_{k=0}^{\infty} (\alpha_w\alpha)^k \Psi_{t,t+k} \left[\frac{\sigma-1}{\sigma}(1+\tau)\bar{\pi}^k P_t(f) - \frac{\bar{\pi}^k W(f)}{MPL_{t+k}(f)}\right] Y_{t+k}(f) = 0. \tag{10}$$

Note that the only difference between (10) and equation (8) in Erceg et al. (2000) is that the probability of an unchanged price is $\alpha_w\alpha$.

To derive labor demand elasticity, first note that we have

$$\frac{dP_t(f)}{dW(f)} = -\frac{\digamma_W}{\digamma_p} = \frac{P_t(f)}{W(f)}. \tag{11}$$

[1] Note that, from (6) and (7) it follows that

$$MC_t(f) = \frac{W_t(f)}{MPL_t(f)}.$$

For future reference, note that from (5) it follows that

$$\frac{\partial L_t(f)}{\partial W(f)} = \frac{\partial L_t(f)}{\partial Y_t(f)} \frac{\partial Y_t(f)}{\partial W(f)} = \frac{1}{1-\gamma} \frac{L_t(f)}{Y_t(f)} \frac{\partial Y_t(f)}{\partial W(f)}, \tag{12}$$

where $\frac{\partial Y_t(f)}{\partial W_t(f)} = \frac{\partial Y_t(f)}{\partial P_t(f)} \frac{\partial P_t(f)}{\partial W_t(f)}$ and thus, using (11) and (3) we have that,

$$\frac{\partial L_t(f)}{\partial W(f)} = -\frac{\sigma}{1-\gamma} \frac{L_t(f)}{W(f)}. \tag{13}$$

The wage elasticity of labor demand is given by

$$\varepsilon_L = \frac{\partial L_t(f)}{\partial W(f)} \frac{W(f)}{L_t(f)} = -\frac{\sigma}{1-\gamma}. \tag{14}$$

2.4 Households

The economy is populated by a representative family, consisting of a continuum of households indexed by h on the unit interval. Moreover, each household is linked to a local labor market with a single firm f. Thus, $h = f$. Each household, in turn, has a continuum of members where a fraction is employed by the firm in the local labor market. Since the family pool income across members, the households are homogeneous with respect to consumption and real money balances. The payoff of having a household member at firm f working is, given the commuting cost j is

$$u(C_t, Q_t) + l\left(\frac{M_t}{P_t}\right) - v(H^e, Z_t) - \vartheta(j)$$

and unemployed is

$$u(C_t, Q_t) + l\left(\frac{M_t}{P_t}\right) - v(0, Z_t)$$

work. The expected life-time utility of the family in period t, is given by

$$E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[u(C_s, Q_s) + l\left(\frac{M_s}{P_s}\right) - \int_0^1 \left(\int_0^{L_s(f)} \left(v(H^e, Z_s) + \vartheta(j) \right) dj \right) df \right.\right. \tag{15}$$

$$\left.\left. - \int_0^1 \int_{L_s(f)}^1 v(0, Z_t)\, dj df \right] \right\}, \tag{16}$$

where $\beta \in (0,1)$ is the households discount factor and $L_s(f)$ is the employment at firm f. Here, C_s is final goods consumption in period s, $\frac{M_s}{P_s}$ is real money balances, where M_s denotes money holdings, and $v(H^e, Z_s)$ and $v(0, Z_s)$ the disutility of being employed and unemployed, respectively. Also, Q_s and Z_s are shocks to the utility of consumption and leisure, respectively. Moreover, there is a distribution over the disutility of supplying labor, ϑ, for each household within the family (due to e.g.

the dislike and distance of commuting) where the household always allocates the household member with the least cost to the labor market giving rise to the term $\vartheta(j)$.[2]

The budget constraint of the family is given by

$$\frac{B_t}{P_t I_t} + \frac{M_t}{P_t} + C_t = \frac{M_{t-1} + B_{t-1}}{P_t} + D_t, \tag{17}$$

where

$$D_t = (1+\tau_w) \int_0^1 \frac{W_t(f) L_t(f)}{P_t} df + \left(1 - \int_0^1 L_t(f)\, df\right) b + \frac{\Gamma_t}{P_t} + \frac{T_t}{P_t} \tag{18}$$

and where I_t is the one period nominal interest rate, B_t denotes one period bonds. Moreover, $W_t(f)$ denotes the households nominal wage and τ_w is the tax rate (subsidy) on labor income. The family owns an equal share of all firms and of the aggregate capital stock. Then, Γ_t is the family's aliquot share of profits and rental income. Also, T_t denotes nominal lump sum transfers from the government. Finally, note that $1 - \int_0^1 L_t(f)\, df$ is equal to the unemployment rate and b is the real monetary payoff to unemployed workers.

The value function corresponding to the family maximization problem is

$$\begin{aligned} \tilde{V}(B_{t-1}, M_{t-1}) = \max \quad & E_t \left\{ u(C_t, Q_t) + l\left(\frac{M_t}{P_t}\right) - \int_0^1 \left(\int_0^{L_t(f)} \left(v(H^e, Z_t) + \vartheta(j) \right) dj \right) df \right. \\ & \left. - \int_0^1 \int_{L_t(f)}^1 v(0, Z_t)\, dj df + \beta \tilde{V}(B_t, M_t) \right\}, \end{aligned} \tag{19}$$

subject to

$$\frac{B_t}{P_t I_t} + \frac{M_t}{P_t} + C_t = \frac{M_{t-1} + B_{t-1}}{P_t} + D_t. \tag{20}$$

Defining

$$V(L_t(h), Z_t) = \left(L_t(f)\, v(H^e, Z_t) + (1 - L_t(f))\, v(0, Z_t) \right) + \left(\int_0^{L_t(f)} \vartheta(j)\, dj \right),$$

we can write

$$\tilde{V}(B_{t-1}, M_{t-1}) = \max E_t \left\{ u(C_t, Q_t) + l\left(\frac{M_t}{P_t}\right) - \int_0^1 V(L_t(f))\, df + \beta \tilde{V}(B_t, M_t) \right\}. \tag{21}$$

[2] An alternative interpretation is given in Cho & Cooley (1994). When unemployed, there is a household production opportunity available for the household. There is a loss $\vartheta(j)$ when a household member j participates in the labor force (or works a fraction $L_s(h)$ of the total available days). Due to decreasing returns in home production this loss increases in $L_s(h)$.

The Lagrangian is then

$$\begin{aligned} L &= E_t\left\{u\left(C_t,Q_t\right)+l\left(\frac{M_t}{P_t}\right)-\int_0^1 V\left(L_t\left(f\right)\right)df+\beta\tilde{V}\left(B_t,M_t\right)\right\} \\ &\quad -\lambda_t\left(\frac{B_t}{P_tI_t}+\frac{M_t}{P_t}+C_t-\frac{M_{t-1}+B_{t-1}}{P_t}-D_t\right). \end{aligned} \tag{22}$$

Using the envelope theorem to compute V_M and V_B and the first-order conditions with respect to C_t and B_t to derive the Euler equation, we arrive at the following expressions

$$l_{\frac{M}{P}}\left(\frac{M_s}{P_s}\right) = \lambda_t\frac{1}{P_t}-\beta E_t\left(u_C\left(C_{t+1},Q_{t+1}\right)\frac{1}{P_{t+1}}\right), \tag{23}$$

$$u_C\left(C_t,Q_t\right) = \beta E_t\left(u_C\left(C_{t+1},Q_{t+1}\right)R_t\right), \tag{24}$$

where

$$R_t = I_t\frac{P_t}{P_{t+1}} \tag{25}$$

is defined as the gross risk-free interest real rate.

For further use, let ρ_C and ρ_L denote the (inverse of the) intertemporal elasticity of substitution in consumption and labor supply, respectively. That is

$$\rho_C = -\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C}, \tag{26}$$

$$\frac{1}{\rho_L} = -\frac{\bar{V}_{LL}\bar{L}}{\bar{V}_L}, \tag{27}$$

where $\bar{x}$ refers to the steady state value of the variable x.

The value of not being on strike for household h is in consumption terms, using (22), the value of the Lagrange multiplier λ_t in optimum and the budget constraint

$$-\frac{V\left(L_t\left(h\right)\right)}{u_C\left(C_t,Q_t\right)}-\left(-\left(\left(1+\tau_w\right)\frac{W_t\left(h\right)}{P_t}L_t\left(h\right)-b\left(1-L_t\left(h\right)\right)\right)\right). \tag{28}$$

2.4.1 Wages

The value for the worker/union at firm f in period $t+k+j$ is given by, given that prices were last changed in $t+k$ and wages in t is denoted $U^t_{t+k,t+k+j}$. Letting $\Upsilon^t_{t+k,t+k+j}$ denote per-period utility, we can write

$$U^t_{t,t} = \Upsilon^t_{t,t}+\alpha_w\beta E_t\frac{u_{C,t+1}}{u_{C,t}}\left(\alpha U^t_{t,t+1}+\left(1-\alpha\right)U^t_{t+1,t+1}\right)+\left(1-\alpha_w\right)\beta E_t\frac{u_{C,t+1}}{u_{C,t}}U^{t+1}_{t+1,t+1}, \tag{29}$$

where

$$\Upsilon_{t+k,t+k+j}^{t} = L_{t+k,t+k+j}^{t}(f)\left((1+\tau_w)\frac{(\bar{\pi})^{k+j}W_t(f)}{P_{t+k+j}}\right)+\left(1-L_{t+k,t+k+j}^{t}(f)\right)b - \frac{V\left(L_{t+k,t+k+j}^{t}(f),Z_{t+k+j}\right)}{u_{C,t+k+j}}. \tag{30}$$

In case the firm and workers renegotiate the wage, bargaining takes place according to the noncooperative Rubinstein-Ståhl model. In case there is disagreement, there is a conflict during the remainder of the period, whereafter negotiations continue in the next period. The payoff in case there is a conflict is then

$$U_{o,t} = \Upsilon_{o,t} + \beta E_t \frac{u_{C,t+1}}{u_{C,t}} U_{u,t+1,t+1}^{t+1}, \tag{31}$$

where

$$\Upsilon_{o,t} = b - \frac{v\left(0, Z_{t+k+j}\right)}{u_{C,t+k+j}}. \tag{32}$$

Some algebra shows that

$$\begin{aligned} U_{t,t}^{t} - U_{o,t} = {} & \Upsilon_{t,t}^{t} - \Upsilon_{o,t} + \sum_{k=1}^{\infty}\left(\alpha_w\alpha\beta\right)^k E_t\frac{u_{C,t+k}}{u_{C,t}}\left(\Upsilon_{t,t+k}^{t}-\Upsilon_{t+1,t+k}^{t}\right) \\ & +\alpha_w\beta\sum_{k=0}^{\infty}\left(\alpha_w\alpha\beta\right)^k E_t\frac{u_{C,t+k+1}}{u_{C,t}}\left(\Upsilon_{t+1,t+k+1}^{t}-\Upsilon_{t+1,t+k+1}^{t+1}\right) \\ & +(1-\alpha)\sum_{k=2}^{\infty}\left(\alpha_w\beta\right)^k\sum_{j=0}^{\infty}\left(\alpha_w\alpha\beta\right)^j E_t\frac{u_{C,t+j+k}}{u_{C,t}}\left(\Upsilon_{t+k,t+j+k}^{t}-\Upsilon_{t+k,t+j+k}^{t+1}\right). \end{aligned} \tag{33}$$

Now consider the firm. We have

$$\begin{aligned} F_{t,t}^{t} &= \phi_{t,t}^{t} + \alpha_w\beta E_t\frac{u_{C,t+1}}{u_{C,t}}\left(\alpha F_{t,t+1}^{t} + (1-\alpha)F_{t+1,t+1}^{t}\right) + (1-\alpha_w)\beta E_t\frac{u_{C,t+1}}{u_{C,t}}F_{t+1,t+1}^{t+1}, \\ F_{o,t} &= 0 + \beta E_t\frac{u_{C,t+1}}{u_{C,t}}F_{t+1,t+1}^{t+1}, \end{aligned} \tag{34}$$

where per-period real profit in period $t+k$ when prices last were changed in t is denoted as

$$\phi_{t+k,t+k+j}^{t} = (1+\tau)\frac{P_{t+k}(f)\,\bar{\pi}^j}{P_{t+k+j}}Y_{t+k,t+k+j}^{t}(f) - tc\left(w_{t+k+j}(f), Y_{t+k,t+k+j}^{t}(f)\right). \tag{35}$$

A similar argument as above shows that

$$\begin{aligned}
F_{t,t}^{t}-F_{o,t} &= \phi_{t,t}^{t}+\sum_{k=1}^{\infty}\left(\alpha_{w}\alpha\beta\right)^{k}E_{t}\frac{u_{C,t+k}}{u_{C,t}}\left(\phi_{t,t+k}^{t}-\phi_{t+1,t+k}^{t}\right) \\
&+\alpha_{w}\beta\sum_{k=0}^{\infty}\left(\alpha_{w}\alpha\beta\right)^{k}E_{t}\frac{u_{C,t+k+1}}{u_{C,t}}\left(\phi_{t+1,t+k+1}^{t}-\phi_{t+1,t+k+1}^{t+1}\right) \\
&+\left(1-\alpha\right)\sum_{k=2}^{\infty}\left(\alpha_{w}\beta\right)^{k}\sum_{j=0}^{\infty}\left(\alpha_{w}\alpha\beta\right)^{j}E_{t}\frac{u_{C,t+j+k}}{u_{C,t}}\left(\phi_{t+k,t+j+k}^{t}-\phi_{t+k,t+j+k}^{t+1}\right).
\end{aligned} \tag{36}$$

Wages are determined in bargaining between the firm and the household attached to the firm. Since there is equivalence between the standard non-cooperative approach in Rubinstein (1982) and the Nash bargaining approach, we use the latter method. We let

$$\begin{aligned}
S_{t,t}^{t} &= U_{t,t}^{t}-U_{o,t}, \\
G_{t,t}^{t} &= F_{t,t}^{t}-F_{o,t}.
\end{aligned} \tag{37}$$

To simplify notation, especially in section 4.4 below, we use gradient notation to indicate derivatives. For example, the partial derivative of the above expressions with respect to the wage W are denoted, $\nabla_{W}S_{t,t}^{t}$ and $\nabla_{W}G_{t,t}^{t}$. The wage is then chosen such that is solves the following problem

$$\max_{W(f)}\left(S_{t,t}^{t}\right)^{\varphi}\left(G_{t,t}^{t}\right)^{1-\varphi}, \tag{38}$$

where φ denotes the bargaining power of households.

The first-order condition corresponding to (38) is

$$\varphi G_{t,t}^{t}\nabla_{W}S_{t,t}^{t}+\left(1-\varphi\right)S_{t,t}^{t}\nabla_{W}G_{t,t}^{t}=0. \tag{39}$$

Alternatively, we can write

$$\varphi\nabla_{W}S_{t,t}^{t}+\left(1-\varphi\right)\frac{S_{t,t}^{t}}{G_{t,t}^{t}}\nabla_{W}G_{t,t}^{t}=0. \tag{40}$$

This is our counterpart to equation (16) in Erceg et al. (2000). Here

$$\begin{aligned}
\nabla_{W}S_{t,t}^{t} &= \nabla_{W}U_{t,t}^{t}=\nabla_{W}\Upsilon_{t,t}^{t}+\sum_{k=1}^{\infty}\left(\alpha_{w}\alpha\beta\right)^{k}E_{t}\frac{u_{C,t+k}}{u_{C,t}}\left(\nabla_{W}\Upsilon_{t,t+k}^{t}-\nabla_{W}\Upsilon_{t+1,t+k}^{t}\right) \\
&+\alpha_{w}\beta\sum_{k=0}^{\infty}\left(\alpha_{w}\alpha\beta\right)^{k}E_{t}\frac{u_{C,t+k+1}}{u_{C,t}}\nabla_{W}\Upsilon_{t+1,t+k+1}^{t} \\
&+\left(1-\alpha\right)\sum_{k=2}^{\infty}\left(\alpha_{w}\beta\right)^{k}\sum_{j=0}^{\infty}\left(\alpha_{w}\alpha\beta\right)^{j}E_{t}\frac{u_{C,t+j+k}}{u_{C,t}}\nabla_{W}\Upsilon_{t+k,t+j+k}^{t},
\end{aligned} \tag{41}$$

where, using (13), we have

$$\nabla_W \Upsilon^t_{t+k,t+k+j} = \varepsilon_L \frac{L^t_{t+k,t+k+j}(f)}{W_t(f)} \left((1+\tau_w) \frac{(\bar{\pi})^{k+j} W_t(f)}{P_{t+k+j}} - b - \frac{V_L\left(L^t_{t+k,t+k+j}(f), Z_{t+k+j}\right)}{u_{C,t+k+j}} \right) + \frac{L^t_{t+k,t+k+j}(f)}{W_t(f)} (1+\tau_w) \frac{\bar{\pi}^{k+j} W_t(f)}{P_{t+k+j}}. \tag{42}$$

Finally, $\nabla_W G^t_{t,t}$ can be written as

$$\begin{aligned} \nabla_W G^t_{t,t} &= \nabla_W F^t_{t,t} = \nabla_W \phi^t_{t,t} + \sum_{k=1}^{\infty} (\alpha_w \alpha \beta)^k E_t \frac{u_{C,t+k}}{u_{C,t}} \left(\nabla_W \phi^t_{t,t+k} - \nabla_W \phi^t_{t+1,t+k} \right) \\ &\quad + \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k E_t \frac{u_{C,t+k+1}}{u_{C,t}} \nabla_W \phi^t_{t+1,t+k+1} \\ &\quad + (1-\alpha) \sum_{k=2}^{\infty} (\alpha_w \beta)^k \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j E_t \frac{u_{C,t+j+k}}{u_{C,t}} \nabla_W \phi^t_{t+k,t+j+k}, \end{aligned} \tag{43}$$

where

$$-\nabla_W \phi^t_{t,t+k} = \frac{tc\left(w_t(f), Y_t(f)\right)}{W(f)} \tag{44}$$

where we have used that the envelope theorem implies that all effects of a change in $W(f)$ on prices are eliminated.

2.5 Steady state

We now turn to the (non-stochastic) steady state of the model.[3] Note that the steady state of the real variables is the same in the flexible price model and the sticky price model. In the steady state, R, C, $Y(f)$ and B are constant. Moreover, $B = 0$. Also, M and P grows with the rate $\bar{\pi}$, i.e., we have $\frac{P_{t+1}}{P_t} = \bar{\pi}$ and $\bar{I} = \bar{R}\bar{\pi}$.

2.5.1 Real wages

At an efficient equilibrium real wages are

$$\bar{w} = \overline{MPL} \tag{45}$$

and, by the resource constraint, we have

$$\bar{Y}(f) = \bar{Y} = \bar{C}. \tag{46}$$

[3] That is, a situation where the disturbances Z_t, Q_t and A_t are equal to their mean values at all dates.

2.5.2 Prices

In steady state, the first-order condition of the firm (10) for price setting becomes

$$(1+\tau)\frac{\sigma-1}{\sigma}-\frac{\bar{w}}{\overline{MPL}}=0, \tag{47}$$

where $\bar{w}$ is the steady state real wage. Since we assume that monetary policy is used only to stabilize deviations from the flexible-price equilibrium, using (45) we require that τ is determined such that $(1+\tau)\frac{\sigma-1}{\sigma}=1$, i.e.

$$\tau=\frac{\sigma}{\sigma-1}-1=\frac{1}{\sigma-1}. \tag{48}$$

2.5.3 Wages

Now, let us turn to the Nash bargaining solution in steady state. The first-order condition (39) then is

$$\varphi\left(\bar{G}\left(W\left(f\right)\right)\right)\nabla_W\bar{S}\left(W\left(f\right)\right)+\left(1-\varphi\right)\left(\bar{S}\left(W\left(f\right)\right)\right)\nabla_W\bar{G}\left(W\left(f\right)\right), \tag{49}$$

where $\bar{S}\left(W\left(f\right)\right)$ etc., indicates that all variables except $W\left(f\right)$ are at steady state levels, noting that the steady state value of $\psi_{t,t+k}$ is $\bar{\psi}_k=\beta^k$. Using (13), (42), (44) and that the real total cost is, using $\overline{mc}=1$, $\frac{1}{1-\gamma}\overline{tc}=\bar{Y}$ gives that,

$$\begin{aligned}\bar{\Upsilon} &= \bar{L}\left(1+\tau_w\right)\bar{w}+\left(1-\bar{L}\right)b-\frac{V\left(\bar{L},\bar{Z}\right)}{\bar{u}_C},\\ \bar{\phi} &= \left(1+\tau\right)\bar{Y}-\overline{tc}=\left(1+\tau\right)\bar{Y}-\left(1-\gamma\right)\bar{Y},\\ \bar{\Upsilon}_o &= b-\frac{V\left(0,\bar{Z}\right)}{\bar{u}_C}.\end{aligned} \tag{50}$$

Then expression (49) can be written as

$$\begin{aligned}&\varphi\left(\tau+\gamma\right)\left(\varepsilon_L\frac{\bar{L}}{W\left(f\right)}\left(\left(1+\tau_w\right)\bar{w}-b\right)-\varepsilon_L\frac{\bar{L}}{W\left(f\right)}\frac{\bar{V}_L}{\bar{u}_C}+\frac{\bar{L}}{W\left(f\right)}\left(1+\tau_w\right)\bar{w}\right)\\ &-\left(1-\varphi\right)\left(\bar{L}\left(\left(1+\tau_w\right)\bar{w}-b\right)-\left(\frac{V\left(\bar{L},\bar{Z}\right)}{\bar{u}_C}-\frac{V\left(0,\bar{Z}\right)}{\bar{u}_C}\right)\right)\left(1-\gamma\right).\end{aligned} \tag{51}$$

Note that, when computing V, we assume $\vartheta\left(j\right)=j^{\varsigma}$ and hence

$$\int_0^{\bar{L}}\left(\vartheta\left(j\right)\right)dj=\frac{\bar{L}^{1+\varsigma}}{1+\varsigma}. \tag{52}$$

2.5.4 Taxes and subsidies

In the paper, we adjust the labor tax/subsidy τ_w, so that efficiency is achieved. From an efficient consumption-leisure choice we have

$$\bar{V}_L = \bar{u}_C \bar{w}. \tag{53}$$

Solving for $1 + \tau_w$ from the first-order condition (51) gives

$$1 + \tau_w = \frac{\varphi \varepsilon_L \left(\frac{b}{\bar{w}} + 1\right) + (1 - \varphi)\left(\frac{b}{\bar{w}} + \frac{V\left(\bar{L}, \bar{Z}\right)}{\bar{u}_C \bar{w} \bar{L}} - \frac{V\left(0, \bar{Z}\right)}{\bar{u}_C \bar{w} \bar{L}}\right) \frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}{\varphi\left(1 + \varepsilon_L\right) + (1 - \varphi) \frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}. \tag{54}$$

2.5.5 Interest

From the Euler equation we get that

$$1 = \frac{1}{\beta} \frac{1}{\bar{I}} \bar{\pi}, \tag{55}$$

or, in real terms, $\bar{R} = \frac{1}{\beta}$ and the nominal interest rate is then $\bar{I} = \frac{\bar{\pi}}{\beta}$.

2.5.6 Equilibrium

We have the following equations that determine the real variables in equilibrium of the economy in steady state. First, efficient consumption-labor choice implies that

$$\frac{\bar{V}_L}{\bar{u}_C} = (1 - \gamma) \frac{\bar{Y}}{\bar{L}}. \tag{56}$$

From efficiency on labor market we have

$$\overline{MPL} = (1 - \gamma) \frac{\bar{Y}}{\bar{L}} = \bar{w}. \tag{57}$$

The reason why (51) does not enter in the two expressions above is that τ_w is used to ensure that the wage bargain leads to an efficient outcome. Second, from the definition of marginal cost

$$\overline{mc} = \frac{\bar{w}}{\overline{MPL}} = 1. \tag{58}$$

Since $\bar{Z}$, $\bar{Q}$, $\bar{A}$, γ and Γ are parameters of the problem, we have six equations and six unknowns. Then, using $\bar{C} = \bar{Y}$ in (56) and (57) with the production technology $\bar{Y} = \bar{A}\left(\bar{L}\right)^{1-\gamma}$ gives the following equation to determine $\bar{L}$

$$\bar{V}_L\left(\bar{L}, \bar{Z}\right) = \bar{u}_C\left(\bar{A}\left(\bar{L}\right)^{1-\gamma}, \bar{Q}\right)\left(\frac{\bar{A}\left(\bar{L}\right)^{1-\gamma}}{\bar{L}}(1 - \gamma)\right). \tag{59}$$

In terms of the function $u = \frac{\left(\bar{C}-\bar{Q}\right)^{1-\chi_C}}{1-\chi_C}$ we have $u_C = \left(\bar{C}-\bar{Q}\right)^{-\chi_C}$. Moreover, we have

$$\rho_C = -\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C} = -\frac{-\chi_C\left(\bar{C}-\bar{Q}\right)^{-\chi_C-1}\bar{C}}{\left(\bar{C}-\bar{Q}\right)^{-\chi_C}} = \chi_C\frac{\bar{C}}{\bar{C}-\bar{Q}} \iff \bar{Q} = \bar{C} - \frac{\chi_C}{\rho_C}\bar{C}. \tag{60}$$

Again, taxes are determined from (48) and (54).[4]

3 Log-linearizing the flexible price equilibrium

Now, let us log-linearize the model around the steady state. We first do this at the flexible price and wage equilibrium. This is then used to derive the log-linearization for the sticky price and wage equilibrium in terms of deviations from flexible-price variables. Here, we focus on a limiting cashless economy.

Let X^* denote the value of a variable in the flexible-price equilibrium.

3.1 Euler Equation

To find the Euler equation we use the definition (26) and log-linearize expression (24). We get

$$\hat{C}_t^* + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{CC}\bar{C}}\hat{Q}_t = E_t\left(\hat{C}_{t+1}^* + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{CC}\bar{C}}\hat{Q}_{t+1} - \frac{1}{\rho_C}\hat{R}_t^*\right). \tag{61}$$

3.2 Prices, real wages and output

Rewriting problem (9) when $\alpha = 0$, we can find $P_t^*(f)$ by maximizing

$$\max_{P_t(f)} E_t\left((1+\tau)P_t^*(f)Y_t^*(f) - TC_t^*(f)\right) \tag{62}$$

$$st.\ Y_t^*(f) = \left(\frac{P_t^*(f)}{P_t^*}\right)^{-\sigma} Y_t^*.$$

Using that $\frac{\sigma}{(1+\tau)(\sigma-1)} = 1$, that $MC_t^* = \frac{W_t^*}{MPL_t^*}$, and that firms choose the same prices and face the same wages in flexible price equilibrium gives

$$P_t^*(f) = MC_t^* = \frac{W_t^*}{MPL_t^*} \iff \frac{W_t^*}{P_t^*} = MPL_t^*. \tag{63}$$

[4]When the model is calibrated, we do not directly use values for χ_C and ς, since we do not have empirical estimates on these. Instead, we use values of ρ_C and ρ_L together with the steady state value of $\bar{L}$, as well as, values for some other parameters of the model to solve for χ_C and ς.

Log-linearizing gives, using the production function,

$$\hat{w}_t^* = \widehat{mpl}_t^* = \hat{A}_t - \gamma \hat{L}_t^*. \tag{64}$$

Also, log-linearizing the production function $Y_t^* = A_t L_t^* (f)^{1-\gamma}$ gives

$$\hat{L}_t^* = \frac{1}{1-\gamma}\left(\hat{Y}_t^* - \hat{A}_t\right). \tag{65}$$

Then, combining (64) and (65) gives

$$\hat{w}_t^* = \frac{1}{1-\gamma}\hat{A}_t - \frac{\gamma}{1-\gamma}\hat{Y}_t^*. \tag{66}$$

We also have the resource constraint for market output $Y_t^* = C_t^*$ or, in log-linearized form

$$\hat{Y}_t^* = \hat{C}_t^*. \tag{67}$$

3.3 Output and Productivity

Due to the tax scheme and flexible prices and wages we have

$$u_C\left(C_t^*, Q_t\right) MPL_t^* = V_L\left(L_t^*, Z_t\right) \tag{68}$$

in equilibrium.

Using that $L_t^* = (1-\gamma)\frac{Y_t^*}{MPL_t^*}$, log-linearizing and using the resource constraint (67) gives

$$\bar{u}_{CC}\bar{C}\overline{MPL}\hat{Y}_t^* + \bar{u}_{CQ}\bar{Q}\overline{MPL}\hat{Q}_t + \bar{u}_C\overline{MPL}\widehat{mpl}_t^* = \bar{V}_{LZ}\bar{Z}\hat{Z}_t + \bar{V}_{LL}\bar{L}\left(\hat{Y}_t^* - \widehat{mpl}_t^*\right). \tag{69}$$

Thus, we have now $\hat{Y}_t^*$ expressed in terms of shocks and $\widehat{mpl}_t^*$. Defining

$$\Lambda^* = \bar{u}_C\left(-\rho_C + \rho_L\frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma}\right) \tag{70}$$

and solving for $\hat{Y}_t^*$ gives

$$\hat{Y}_t^* = \frac{1}{\Lambda^*}\left(-\bar{u}_{CQ}\bar{Q}\hat{Q}_t - \bar{u}_C\frac{1}{1-\gamma}\hat{A}_t + \frac{\bar{V}_{LZ}\bar{Z}}{\bar{w}}\hat{Z}_t + \frac{\bar{V}_{LL}\bar{L}}{\bar{w}}\left(-\frac{1}{1-\gamma}\hat{A}_t\right)\right). \tag{71}$$

3.4 Wages

Recall that the wage is chosen to solve (38). Note that

$$\begin{aligned} G^t_{t,t} &= \left((1+\tau) Y^*_t - tc\left(w^*_t, Y^*_t (f)\right)\right), \\ S^t_{t,t} &= L^*_t \left((1+\tau_w) w^*_t - b\right) - \left(\frac{V\left(L^*_t, Z_t\right)}{u_C\left(C^*_t, Q^*_t\right)} - \frac{V\left(0, Z_t\right)}{u_C\left(C^*_t, Q^*_t\right)} \right), \end{aligned} \tag{72}$$

and, using (13), gives

$$\begin{aligned} \nabla_W S^t_{t,t} &= \frac{L^*_t}{W^*_t} \left((1+\varepsilon_L)(1+\tau_w) w^*_t - \varepsilon_L b \right) - \varepsilon_L \frac{L^*_t}{W^*_t} \frac{V_L\left(L^*_t, Z_t\right)}{u_C\left(C^*_t, Q^*_t\right)}, \\ \nabla_W G^t_{t,t} &= -\frac{tc\left(w^*_t, Y^*_t\right)}{W^*_t}. \end{aligned} \tag{73}$$

The first-order condition (39) can then be written as

$$\begin{aligned} &\varphi\left((1+\tau) Y^*_t - tc\left(w^*_t, p^{c*}_t, Y^*_t\right)\right) \frac{L^*_t}{W^*_t} \\ &\times \left(\left((1+\varepsilon_L)(1+\tau_w) w^*_t - \varepsilon_L b \right) - \varepsilon_L \frac{V_L\left(L^*_t, Z_t\right)}{u_C\left(C^*_t, Q^*_t\right)} \right) \\ &- (1-\varphi) \left(L^*_t \left((1+\tau_w) w^*_t - b\right) - \left(\frac{V\left(L^*_t, Z_t\right)}{u_C\left(C^*_t, Q^*_t\right)} - \frac{V\left(0, Z_t\right)}{u_C\left(C^*_t, Q^*_t\right)} \right) \right) tc\left(w^*_t, Y^*_t\right). \end{aligned} \tag{74}$$

3.5 Interest

The relationship between nominal and real interest rates is derived from $R_t = \frac{P_t}{P_{t+1}} I_t$. We have, using that $\pi_{t+1} = \frac{P_{t+1}}{P_t}$,

$$\hat{I}_t - E_t \hat{\pi}_{t+1} = E_t \hat{R}_t. \tag{75}$$

3.6 Shocks and real wages

Using (26), (27), (66), (69), (70) and that $\hat{w}^*_t = \widehat{mpl}^*_t$ to write $\hat{w}^*_t$ in terms of shocks only gives

$$\hat{w}^*_t = \frac{1}{\Lambda^*} \bar{u}_C \left(-\rho_C + \rho_L\right) \frac{1}{1-\gamma} \hat{A}_t + \frac{1}{\Lambda^*} \frac{\gamma}{1-\gamma} \left(\bar{u}_{CQ} \bar{Q} \hat{Q}_t - \frac{\bar{Z} \bar{V}_{LZ}}{\bar{w}} \hat{Z}_t \right). \tag{76}$$

Since $\Lambda^* < 0$, the coefficient in front of $\hat{A}_t$ is positive. The coefficients in front of $\hat{Q}_t$ and $\hat{Z}_t$ depend on the cross derivative of u and the sign of $v_Z\left(H^e, \bar{Z}\right) - v_Z\left(0, \bar{Z}\right)$. If $\bar{u}_{CQ}$ is positive as in Erceg et al. (2000), the coefficient in front of $\hat{Q}_t$ and $\hat{Z}_t$ are negative. Note that, in terms of the notation in the main text, we have

$$\begin{aligned} a_Q &= \frac{1}{\Lambda^*}\frac{\gamma}{1-\gamma}\bar{u}_{CQ}\bar{Q} < 0, \\ a_Z &= -\frac{1}{\Lambda^*}\frac{\gamma}{1-\gamma}\frac{\bar{Z}}{\bar{w}}\bar{V}_{LZ} < 0, \\ a_A &= \frac{1}{\Lambda^*}\bar{u}_C\left(-\rho_C+\rho_L\right)\frac{1}{1-\gamma} > 0. \end{aligned} \tag{77}$$

To simplify analysis we suppress the shocks $\hat{Q}_t$, $\hat{Z}_t$ and $\hat{A}_t$ and assume that $\hat{w}_t^*$ follows an AR(1) process

$$\hat{w}_t^* = \eta\hat{w}_{t-1}^* + \varepsilon_t. \tag{78}$$

Note that there is no simple relationship between real wages and output, as can be seen by inspecting (66). When we have shocks in $\hat{Q}_t$ and $\hat{Z}_t$ the relationship is simple with $\hat{Y}_t^* = -\frac{1-\gamma}{\gamma}\hat{w}_t^*$. In case we have shocks in $\hat{A}_t$ we instead have

$$\hat{Y}_t^* = \frac{1}{\gamma}\hat{A}_t - \frac{1-\gamma}{\gamma}\hat{w}_t^*. \tag{79}$$

Consider shocks in $\hat{A}_t$ only. We then get

$$\begin{aligned} \hat{w}_t^* &= \frac{1}{\Lambda^*}\bar{u}_C\left(-\rho_C+\rho_L\right)\frac{1}{1-\gamma}\hat{A}_t, \\ \hat{Y}_t^* &= \frac{1-\rho_L}{\rho_C-\rho_L}\hat{w}_t^*. \end{aligned} \tag{80}$$

Since $\rho_C > 0$ we get a positive relationship between output and real wages.

Only productivity shocks leads to a positive relationship between real wages and output. Other shocks lead to a negative relationship. The total effects of shocks on output is then unclear.

Assuming that all shocks have the same persistence η, we get from (61), (65) and (67)

$$\hat{C}_t^* + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{CC}\bar{C}}\hat{Q}_t = E_t\left(\hat{C}_{t+1}^* + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{CC}\bar{C}}\hat{Q}_{t+1} - \frac{1}{\rho_C}\hat{R}_t^*\right). \tag{81}$$

Using (78) and assuming only productivity shocks gives

$$\hat{R}_t^* = -\rho_C\left(1-\eta\right)\frac{1-\rho_L}{\rho_C-\rho_L}\hat{w}_t^*. \tag{82}$$

In general, we get

$$\hat{R}_t^* = -\frac{1-\eta}{\rho_C-\rho_L\frac{1}{1-\gamma}+\frac{\gamma}{1-\gamma}}\left(\rho_C\frac{1-\rho_L}{1-\gamma}\hat{A}_t + \rho_C\frac{\bar{Z}\bar{V}_{LZ}}{\bar{u}_C\bar{w}}\hat{Z}_t + \frac{1-\rho_L}{1-\gamma}\frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_C}\hat{Q}_t\right). \tag{83}$$

4 Log-linearizing the sticky price equilibrium

Now, let us log-linearize the model with sticky prices. As above, we start by log-linearizing the Euler equation.

4.1 Euler and the IS equation

Log-linearizing expression (24) gives,

$$\hat{C}_t - \hat{C}_t^* = E_t \left(\hat{C}_{t+1} - \hat{C}_{t+1}^* - \frac{1}{\rho_C} \left(\hat{I}_t - \hat{\pi}_{t+1} - \hat{R}_t^* \right) \right). \tag{84}$$

Solving for the interest rate gives

$$\hat{I}_t = \rho_C \left(E_t \hat{x}_{t+1} - \hat{x}_t \right) + E_t \hat{\pi}_{t+1} - \rho_C \left(1 - \eta \right) \frac{1 - \rho_L}{\rho_C - \rho_L} \hat{w}_t^*, \tag{85}$$

where

$$\hat{x}_t = \hat{Y}_t - \hat{Y}_t^* \tag{86}$$

denotes the output gap.

4.2 Log-linearization of some real and nominal variables

Before we can proceed to log-linearize the price and wage setting decisions, we need to log-linearize some other variables in the model.

4.2.1 Marginal product

To derive an expression for the marginal product, we first log-linearize the production function as

$$\hat{Y}_t \left(f \right) = \hat{A}_t + \left(1 - \gamma \right) \hat{L}_t \left(f \right). \tag{87}$$

Log-linearizing expression (7) and aggregating over firms gives

$$\widehat{mpl}_t = \hat{A}_t - \gamma \hat{L}_t, \tag{88}$$

where $\hat{L}_t = \int \hat{L}_t \left(f \right) df$ and $\widehat{mpl}_t = \int \widehat{mpl}_t \left(f \right) df$ is the aggregate real wage and marginal product, respectively.

Using (87) in (88) gives

$$\widehat{mpl}_t = \frac{1}{1 - \gamma} \left(\hat{A}_t - \gamma \hat{Y}_t \right). \tag{89}$$

To derive a relationship between $\widehat{mpl}_t$ and the (flexible price) real wage, we use expression (66) and hence we get

$$\widehat{mpl}_t = \hat{w}_t^* - \frac{\gamma}{1-\gamma}\hat{x}_t. \tag{90}$$

4.2.2 Marginal rate of substitution

The marginal rate of substitution is defined as

$$MRS_t = \frac{V_L(L_t, Z_t)}{u_C(C_t, Q_t)}. \tag{91}$$

Log-linearizing and integrating over all firms/unions, using that we from expressions (56) and (57) have $\overline{MPL} = \overline{MRS}$ and that $\hat{C}_t = \hat{Y}_t$, gives

$$\widehat{mrs}_t = -\rho_L \hat{L}_t + \rho_C \hat{C}_t - \frac{\bar{u}_{CQ}}{\bar{u}_C}\bar{Q}\hat{Q}_t + \frac{\bar{V}_{LZ}}{\bar{u}_C \bar{w}}\bar{Z}\hat{Z}_t. \tag{92}$$

Subtracting flexible-price marginal rate of substitution gives

$$\widehat{mrs}_t - \widehat{mrs}_t^* = -\rho_L\left(\hat{L}_t - \hat{L}_t^*\right) + \rho_C\left(\hat{Y}_t - \hat{Y}_t^*\right). \tag{93}$$

Using the production function and integrating over all firms gives

$$\hat{Y}_t - \hat{Y}_t^* = (1-\gamma)\left(\hat{L}_t - \hat{L}_t^*\right). \tag{94}$$

Using $\widehat{mrs}_t^* = \widehat{mpl}_t^* = \hat{w}_t^*$ and (94), expression (93) can be rewritten as,

$$\widehat{mrs}_t = \hat{w}_t^* + \left(\rho_C - \rho_L\frac{1}{1-\gamma}\right)\hat{x}_t. \tag{95}$$

4.2.3 Relative prices and goods demand

We define the firms relative prices and wages as

$$\begin{aligned} q_t(f) &= \frac{P_t(f)}{P_t}, \\ n_t(f) &= \frac{W(f)}{W_t}, \\ w_t(f) &= \frac{W(f)}{P_t} \end{aligned} \tag{96}$$

and also

$$X_{t,k} = \frac{\bar{\pi}^k P_t}{P_{t+k}} = \frac{\bar{\pi}^k}{\pi_{t+1} \cdot \ldots \cdot \pi_{t+k}}, \tag{97}$$
$$X_{t,k}^{\omega} = \frac{\bar{\pi}^k W_t}{W_{t+k}} = \frac{\bar{\pi}^k}{\pi_{t+1}^{\omega} \cdot \ldots \cdot \pi_{t+k}^{\omega}}.$$

The real wage at firm f at time $t+k$ is

$$\frac{W(f)\,\bar{\pi}^k}{P_{t+k}} = \frac{W(f)\,\bar{\pi}^k}{W_{t+k}} w_{t+k} = n_t(f)\, X_{t,k}^{\omega} w_{t+k}. \tag{98}$$

4.2.4 Output, labor demand, costs and profits

Loglinearizing (87) we can write

$$\hat{Y}_t(f) = \hat{A}_t + (1-\gamma)\,\hat{L}_t(f)\,. \tag{99}$$

Consider the derivative of labor demand (13). Log-linearizing gives

$$\widehat{\frac{\partial L_{t,t+k}(f)}{\partial W(f)}} = \hat{L}_{t,t+k}(f)\,. \tag{100}$$

Log-linearizing goods demand, using (3) and (96), gives

$$\hat{Y}_{t+k}(f) = -\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \hat{Y}_{t+k}. \tag{101}$$

Log-linearizing total costs (8), using a log-linearization of (96) and (98), we get

$$\widehat{tc}\left(\frac{W(f)\,\bar{\pi}^k}{P_{t+k}}, Y_{t+k}(f)\right) = \left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}^{\omega} + \hat{w}_{t+k}\right) - \frac{1}{1-\gamma}\hat{A}_{t+k} \tag{102}$$
$$-\sigma\frac{1}{1-\gamma}\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\hat{Y}_{t+k}.$$

To log-linearize labor demand, using goods demand (5) we have

$$\hat{L}_{t,t+k}(f) = -\sigma\frac{1}{1-\gamma}\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\hat{Y}_{t+k} - \frac{1}{1-\gamma}\hat{A}_{t+k}. \tag{103}$$

Total costs can be rewritten, using (103) and goods demand, as

$$\widehat{tc}\left(\frac{W\left(f\right)\bar{\pi}^{k}}{P_{t+k}},Y_{t+k}\left(f\right)\right)=\hat{L}_{t,t+k}\left(f\right)+\left(\hat{n}_{t}\left(f\right)-\sum_{l=1}^{k}\hat{\pi}_{t+l}^{\omega}\right)+\hat{w}_{t+k}. \tag{104}$$

The log-linearized version of per period profits (35) is, using (101), (103) and (104),

$$\begin{aligned}\bar{\phi}\hat{\phi}_{t,t+k} &= (1+\tau)\bar{Y}(1-\sigma)\left(\hat{q}_{t}(f)-\sum_{l=1}^{k}\hat{\pi}_{t+l}\right)+(1+\tau)\bar{Y}\hat{Y}_{t+k}-\overline{tc}\left(-\frac{\sigma}{1-\gamma}\left(\hat{q}_{t}(f)-\sum_{l=1}^{k}\hat{\pi}_{t+l}\right)\right.\\ &\left.+\frac{1}{1-\gamma}\left(\hat{Y}_{t+k}-\hat{A}_{t+k}\right)+\left(\hat{n}_{t}(f)-\sum_{l=1}^{k}\hat{\pi}_{t+l}^{\omega}\right)+\hat{w}_{t+k}\right). \end{aligned}\tag{105}$$

Also, we have

$$\nabla_{W}\phi_{t,t+k}=-\frac{1}{P_{t+k}}\frac{\partial TC\left(\bar{\pi}^{k}W\left(f\right),Y_{t+k}\left(f\right)\right)}{\partial W\left(f\right)}=-\frac{tc\left(\frac{W(f)\bar{\pi}^{k}}{P_{t+k}},Y_{t+k}\left(f\right)\right)}{W\left(f\right)}. \tag{106}$$

We then get the log-linearized version of the derivative of per-period profits as

$$\begin{aligned}-\widehat{\nabla_{W}\phi}_{t,t+k} &= \widehat{tc}_{t,t+k}=-\frac{\sigma}{1-\gamma}\left(\hat{q}_{t}(f)-\sum_{l=1}^{k}\hat{\pi}_{t+l}\right)+\frac{1}{1-\gamma}\left(\hat{Y}_{t+k}-\hat{A}_{t+k}\right)\\ &+\left(\hat{n}_{t}(f)-\sum_{l=1}^{k}\hat{\pi}_{t+l}^{\omega}\right)+\hat{w}_{t+k}.\end{aligned}\tag{107}$$

4.2.5 Total demand and unemployment

The log-linear approximation of total demand is

$$\hat{Y}_{t}=\hat{C}_{t}. \tag{108}$$

Using the production function gives, integrating over all firms,

$$\hat{Y}_{t}=\hat{A}_{t}+(1-\gamma)\hat{L}_{t}. \tag{109}$$

Thus, we get

$$\hat{x}_{t}=\hat{A}_{t}+(1-\gamma)\hat{L}_{t}-\hat{Y}_{t}^{*} \tag{110}$$

and hence, using (80) and that unemployment is $u_t = 1 - L_t$

$$\hat{u}_{t}=-\frac{\bar{L}}{1-\bar{L}}\hat{L}_{t}=-\frac{\bar{L}}{1-\bar{L}}\left(\frac{1}{1-\gamma}\hat{x}_{t}+\frac{(1-\rho_{C})}{\rho_{C}-\rho_{L}}\hat{w}_{t}^{*}\right). \tag{111}$$

4.2.6 Real wage evolution

The real wage today can be written as a function of the previous period real wage as follows

$$\frac{W_t}{P_t} = \frac{\pi_t^{\omega} W_{t-1}}{\pi_t P_{t-1}}. \tag{112}$$

Log-linearizing gives

$$\hat{w}_t = \hat{w}_{t-1} + \hat{\pi}_t^{\omega} - \hat{\pi}_t. \tag{113}$$

4.3 Optimal Prices and the New Keynesian Phillips curve

The first-order condition for optimal price choices, i.e. expression (10), can be rewritten as

$$E_t \sum_{k=0}^{\infty} (\alpha_w \alpha)^k \Psi_{t,t+k} \left(q_t(f) X_{t,k} - mc_{t+k}(f)\right) Y_{t+k}(f) = 0, \tag{114}$$

where $mc_{t+k}(f)$ is the real marginal cost. Log-linearizing around steady state, and using that $\bar{\Psi}_k = P_t (\bar{\pi}\beta)^k$ (the value of the steady state path of $\Psi_{t,t+k}$, given an initial price level P_t), that $P_t \neq 0$ and that the probability that wages not open for renegotiation in period $t+k$ is $\alpha_w \alpha$ gives [5]

$$0 = E_t \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \left(\hat{q}_t(f) + \hat{X}_{t,k} - \widehat{mc}_{t+k}(f)\right) \bar{Y}. \tag{115}$$

Now, let us derive the aggregate supply equation (i.e., new Keynesian Phillips curve). Log-linearizing $X_{t,k}$ in (97) gives

$$-\sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k E_t \hat{X}_{t,k} = \sum_{l=1}^{\infty} \frac{(\alpha_w \alpha \beta)^l}{1 - \alpha_w \alpha \beta} E_t \hat{\pi}_{t+l}. \tag{116}$$

Note that the wage distribution of the firms that change prices is not the same as for the entire population of firms. Let W_t^o denote the solution to problem (38). The average wage for those firms that change prices is then

$$W_t^p = \frac{(1-\alpha)\alpha_w}{(1-\alpha)\alpha_w + (1-\alpha_w)} \int \bar{\pi} W_{t-1}(f)\, df + \frac{(1-\alpha_w)}{(1-\alpha)\alpha_w + (1-\alpha_w)} \int W_t^o df. \tag{117}$$

The entire wage distribution evolves according to

$$W_t = \alpha_w \int \bar{\pi} W_{t-1}(f)\, df + (1-\alpha_w) \int W_t^o df. \tag{118}$$

[5] Note that $\Psi_{t,t+k} = \psi_{t,t+k} P_{t+k}$. Also, we have $\psi_{t,t+k} = \prod_{s=1}^{k} \psi_{t+s-1,t+s}$. Also, we normalize $\psi_{t,t} = 1$ as in Woodford (2003) page 68.

Using (118) in (117), we get, in real terms

$$w_t^p = \frac{w_t}{(1-\alpha)\alpha_w + (1-\alpha_w)} \left(1 - \alpha_w \alpha \frac{\bar{\pi}}{\pi_t^\omega}\right). \tag{119}$$

Log-linearizing (119), evaluating the real wage in $t+k$, and hence taking into account the effects of inflation on the real wage through $\hat{X}_{t,k}$ gives

$$\hat{w}_{t+k}^p = \hat{w}_t + \frac{\alpha_w \alpha}{1-\alpha_w \alpha} \hat{\pi}_t^\omega + \hat{X}_{t,k}. \tag{120}$$

Deriving real marginal cost from the total cost expression in (6), (101) and log-linearizing gives

$$\widehat{mc}_{t+k}(f) = \hat{w}_{t+k}^p(f) + \left(-\sigma \frac{\gamma}{1-\gamma}\left(\hat{q}_t(f) - \sum_{l=1}^{k} \hat{\pi}_{t+l}\right) + \frac{\gamma}{1-\gamma}\hat{Y}_{t+k} - \frac{1}{1-\gamma}\hat{A}_{t+k}\right), \tag{121}$$

where $\hat{w}_{t+k}^p(f)$ is the log-linearized real wage for firms that change prices in t. Note that the average marginal cost for firms that change prices is, using the above expression (121) and expression (120)

$$\widehat{mc}_{t+k} = \left(\hat{w}_t + \frac{\alpha_w \alpha}{1-\alpha_w \alpha}\hat{\pi}_t^\omega + \hat{X}_{t,k}\right) + \left(-\sigma \frac{\gamma}{1-\gamma}\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \frac{\gamma}{1-\gamma}\hat{Y}_{t+k} - \frac{1}{1-\gamma}\hat{A}_{t+k}\right). \tag{122}$$

Expression (115) can be rewritten, aggregating over all firms that change prices and using (116)

$$\begin{aligned} 0 &= \frac{1}{1-\alpha_w \alpha \beta}\left(\hat{q}_t\left(1 + \sigma \frac{\gamma}{1-\gamma}\right) - \left(\hat{w}_t + \frac{\alpha_w \alpha}{1-\alpha_w \alpha}\hat{\pi}_t^\omega\right)\right) \\ &\quad - E_t \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \frac{1}{1-\gamma}\left(\gamma \hat{Y}_{t+k} - \hat{A}_{t+k}\right) - \sigma \frac{\gamma}{1-\gamma}\sum_{k=1}^{\infty} \frac{(\alpha_w \alpha \beta)^k}{1-\alpha_w \alpha \beta} E_t \hat{\pi}_{t+k}. \end{aligned} \tag{123}$$

To write the expression above in terms of inflation, we need to express the relative prices $\hat{q}_t(f)$ in terms of inflation. To do this, we use the price evolution equation. Using that prices evolve according to

$$P_t^{1-\sigma} = \alpha_w \alpha \int_0^1 (\bar{\pi} P_{t-1}(f))^{1-\sigma} df + (1-\alpha_w \alpha)\int_0^1 (P_t^o(f))^{1-\sigma} df, \tag{124}$$

gives, using $\frac{P_{t-1}}{P_t} = \frac{1}{\pi_t}$

$$1 = \alpha_w \alpha \left(\frac{\bar{\pi}}{\pi_t}\right)^{1-\sigma} + (1-\alpha_w \alpha)\int (q_t(f))^{1-\sigma} df. \tag{125}$$

We thus have that

$$\hat{q}_t = \int \hat{q}_t(f)\, df = \frac{\alpha_w \alpha}{1-\alpha_w \alpha}\hat{\pi}_t. \tag{126}$$

The first-order condition for price setting (115) can then be rewritten by using (126) in (123) in periods

t and $t+1$, respectively, together with the real wage identity (113),

$$\begin{aligned} 0 \quad = \quad & \left(\frac{\alpha_w\alpha}{1-\alpha_w\alpha}\hat{\pi}_t\left(1+\sigma\frac{\gamma}{1-\gamma}\right)-\left(\frac{\alpha_w\alpha}{1-\alpha_w\alpha}\left(\hat{\pi}_t^{\omega}-\beta E_t\hat{\pi}_{t+1}^{\omega}\right)\right)\right)-(1-\alpha_w\alpha\beta)\,\hat{w}_t \\ & -(1-\alpha_w\alpha\beta)\,E_t\frac{1}{1-\gamma}\left(\gamma\hat{Y}_t-\hat{A}_t\right)-\left(1+\sigma\frac{\gamma}{1-\gamma}\right)\frac{\alpha_w\alpha}{1-\alpha_w\alpha}\beta E_t\hat{\pi}_{t+1}. \end{aligned} \tag{127}$$

Using (126) gives the result. To eliminate $\hat{Y}_t$ and $\hat{A}_t$ from the above expression, we use (66) to get

$$\gamma\hat{Y}_t-\hat{A}_t=\gamma\hat{x}_t-(1-\gamma)\,\hat{w}_t^{*}. \tag{128}$$

Then, defining

$$\Pi=\frac{1-\alpha_w\alpha}{\alpha_w\alpha}\frac{(1-\alpha_w\alpha\beta)}{1+\sigma\frac{\gamma}{1-\gamma}}, \tag{129}$$

the first-order condition for price setting, or equivalently, the New Keynesian Phillips curve, is

$$\hat{\pi}_t=\beta E_t\hat{\pi}_{t+1}+\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{\pi}_t^{\omega}-\beta E_t\hat{\pi}_{t+1}^{\omega}\right)+\Pi\left(\hat{w}_t-\hat{w}_t^{*}\right)+\frac{\gamma}{1-\gamma}\Pi\hat{x}_t. \tag{130}$$

The only difference with expression (T1.4) in Erceg et al. (2000) is the presence of the term involving wage inflation. Using (90), we can rewrite (130) as

$$\hat{\pi}_t=\beta E_t\hat{\pi}_{t+1}+\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{\pi}_t^{\omega}-\beta E_t\hat{\pi}_{t+1}^{\omega}\right)+\Pi\left(\hat{w}_t-\widehat{mpl}_t\right). \tag{131}$$

4.3.1 Relationship between relative prices and wages

To analyze wage setting, we need to relate the relative prices to relative wages for the price adjusting firms (see the section 4.4 below on wage determination). Let us first look at the relationship between relative prices and wages for firms that changed wages in t and prices in $t+k$. The first order condition for price setting (10) is, where $\hat{q}_{t+k}^{t}$ is the log-linearized relative price in $t+k$ for firms that renegotiated their wages in t, and $\hat{n}^{t}$ the relative wage for firms that renegotiated their wages last in period t

$$0=E_t\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j\left(\hat{q}_{t+k}^{t}+\hat{X}_{t+k,k+j}-\widehat{mc}_{t+k+j}\left(f\right)\right)\bar{Y}, \tag{132}$$

where, deriving marginal cost from the expression (8) for total costs, and using that

$$w_{t+k}\left(f\right)=\frac{\bar{\pi}^k W_t\left(f\right)}{P_{t+k}}=\frac{W_t\left(f\right)}{W_t}\frac{W_t}{W_{t+k}}\frac{W_{t+k}}{P_{t+k}}, \tag{133}$$

we have

$$\widehat{mc}_{t+k+j}(f) = \hat{w}_{t+k+j} + \left(\hat{n}^t + \hat{X}^{\omega}_{t,k+j}\right) - \sigma\frac{\gamma}{1-\gamma}\left(\hat{q}^t_{t+k} + \hat{X}_{t+k,k+j}\right) \tag{134}$$

$$-\left(\frac{1}{1-\gamma}\hat{A}_{t+k+j} - \frac{\gamma}{1-\gamma}\hat{Y}_{t+k+j}\right). \tag{135}$$

Using (66) gives

$$\widehat{mc}_{t+k+j}(f) = \hat{w}_{t+k+j} - \hat{w}^*_{t+k+j} + \left(\hat{n}^t + \hat{X}^{\omega}_{t,k+j}\right) - \sigma\frac{\gamma}{1-\gamma}\left(\hat{q}^t_{t+k} + \hat{X}_{t+k,k+j}\right) \tag{136}$$

$$+\frac{\gamma}{1-\gamma}\hat{x}_{t+k+j}. \tag{137}$$

Rewriting the sums over $\hat{X}_{t+k,k+j}$ and $\hat{X}^{\omega}_{t,k+j}$ in expression (132) gives

$$-\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{X}_{t+k,k+j} = \sum_{l=1}^{\infty}\frac{(\alpha_w\alpha\beta)^l}{1-\alpha_w\alpha\beta}E_t\hat{\pi}_{t+k+l}, \tag{138}$$

$$-\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{X}^{\omega}_{t,k+j} = \frac{1}{1-\alpha_w\alpha\beta}\left(\sum_{l=1}^{k}E_t\hat{\pi}^{\omega}_{t+l} + \sum_{l=1}^{\infty}(\alpha_w\alpha\beta)^l E_t\hat{\pi}^{\omega}_{t+k+l}\right).$$

Then, using the expression for $\widehat{mc}_{t+k+j}(f)$ in the first-order condition (132), we have

$$0 = \hat{q}^t_{t+k}\left(1+\sigma\frac{\gamma}{1-\gamma}\right) - \hat{n}^t - (1-\alpha_w\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j\left(\hat{w}_{t+k+j} - \hat{w}^*_{t+k+j} + \frac{\gamma}{1-\gamma}\hat{x}_{t+k+j}\right)$$

$$-\left(1+\sigma\frac{\gamma}{1-\gamma}\right)\sum_{j=1}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{\pi}_{t+k+j} + \left(\sum_{l=1}^{k}E_t\hat{\pi}^{\omega}_{t+l} + \sum_{j=1}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{\pi}^{\omega}_{t+k+j}\right). \tag{139}$$

Leading one wage contract period ahead and combining gives

$$\hat{q}^t_{t+k} - E_t\hat{q}^{t+1}_{t+k} = \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{n}^t - E_t\hat{n}^{t+1} - E_t\hat{\pi}^{\omega}_{t+1}\right). \tag{140}$$

For the analysis of wages below, we also want to derive a relationship between relative prices in t and in $t+1$ for firms that last changed wages in period t. From using (139) when wages are renegotiated at t and prices at t and $t+1$, respectively, we have

$$\hat{q}^t_t - \alpha_w\alpha\beta\left(E_t\hat{q}^t_{t+1} + E_t\hat{\pi}_{t+1}\right) = \frac{(1-\alpha_w\alpha\beta)}{\left(1+\sigma\frac{\gamma}{1-\gamma}\right)}\hat{n}^t + \frac{(1-\alpha_w\alpha\beta)}{\left(1+\sigma\frac{\gamma}{1-\gamma}\right)}\left(\hat{w}_t - \hat{w}^*_t + \frac{\gamma}{1-\gamma}\hat{x}_t\right). \tag{141}$$

4.4 Optimal Wages and the wage setting "Phillips" Curve

In this section we derive the wage setting "Phillips" curve from expression (39). Log-linearizing the first-order condition (39) gives

$$
\begin{aligned}
0 &= \varphi \nabla_W \bar{S} \widehat{\nabla_W S^t_{t,t}} + (1-\varphi)\left(\frac{1}{\bar{G}} \nabla_W \bar{G} \bar{S} \hat{S}^t_{t,t} - \frac{\bar{S}}{(\bar{G})^2} \nabla_W \bar{G} \bar{G} \hat{G}^t_{t,t}\right) \\
&\quad + (1-\varphi) \frac{\bar{S}}{\bar{G}} \nabla_W \bar{G} \widehat{\nabla_W G^t_{t,t}}.
\end{aligned} \tag{142}
$$

4.4.1 Firm $G^t_{t,t}$

We have

$$
\begin{aligned}
&\bar{G}\hat{G}^t_{t,t} = \bar{\phi}\hat{\phi}^t_{t,t} + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha \beta)^k \bar{\phi} \left(\hat{\phi}^t_{t,t+k} - \hat{\phi}^t_{t+1,t+k}\right) \\
&+\alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \bar{\phi} \left(\hat{\phi}^t_{t+1,t+1+k} - \hat{\phi}^{t+1}_{t+1,t+1+k}\right) \\
&+E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^k (1-\alpha) \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \bar{\phi} \left(\hat{\phi}^t_{t+k,t+k+j} - \hat{\phi}^{t+1}_{t+k,t+k+j}\right).
\end{aligned} \tag{143}
$$

where, using (105) and that $(1+\tau)(1-\sigma)\bar{Y} + \frac{\sigma}{1-\gamma}\overline{tc} = 0$,

$$
\begin{aligned}
\bar{\phi}\hat{\phi}^t_{t,t} &= -\overline{tc}\hat{n}^t + R^{f,t}_{t,t}, \\
\bar{\phi}\hat{\phi}^t_{t,t+k} - \bar{\phi}\hat{\phi}^t_{t+1,t+k} &= 0, \\
\bar{\phi}\hat{\phi}^t_{t+k,t+k+j} - \bar{\phi}\hat{\phi}^{t+1}_{t+k,t+k+j} &= -\overline{tc}\left(\hat{n}^t - \hat{n}^{t+1}\right) + \overline{tc}\hat{\pi}^{\omega}_{t+1}.
\end{aligned} \tag{144}
$$

Define

$$
\Delta \hat{n}^t = \frac{1}{1-\alpha_w \beta}\left(\hat{n}^t - \alpha_w \beta \left(E_t \hat{n}^{t+1} + E_t \hat{\pi}^{\omega}_{t+1}\right)\right). \tag{145}
$$

Then, using (145), we have

$$
\bar{G}\hat{G}^t = -\overline{tc}\Delta\hat{n}^t + R^{f,t}_{t,t}, \tag{146}
$$

where, using (66), we have

$$
R^{f,t}_{t,t} = (1+\tau)\bar{Y}\hat{Y}_t - \overline{tc}\left(\hat{w}_t + \left(\hat{Y}_t - \hat{A}_t\right)\frac{1}{1-\gamma}\right). \tag{147}
$$

In terms of differences with the flexible price equilibrium, we can write

$$
R^{f,t}_{t,t} - R^{f,t*}_{t,t} = \left((1+\tau)\bar{Y} - \frac{1}{1-\gamma}\overline{tc}\right)\hat{x}_t - \overline{tc}\left(\hat{w}_t - \hat{w}^*_t\right). \tag{148}
$$

4.4.2 Firm $\nabla_W G^t_{t,t}$

To rewrite $\nabla_W G^t_{t,t}$ note that

$$\begin{aligned}
\nabla_W \bar{G}\left(-\widehat{\nabla_W G^t_{t,t}}\right) &= \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t,t}\right) + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha\beta)^k \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t,t+k} - \left(-\widehat{\nabla_W \phi}^t_{t+1,t+k}\right)\right) \\
&+ \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha\beta)^k \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t+1,t+1+k}\right) \\
&+ E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^{k-1} (\alpha_w - \alpha_w \alpha)\, \beta \sum_{j=0}^{\infty} (\alpha_w \alpha\beta)^j \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t+k,t+k+j}\right).
\end{aligned} \tag{149}$$

Subtracting $\alpha_w \beta \nabla_W \bar{G} E_t \left(-\nabla_W \widehat{G^{t+1}_{t+1,t+1}}\right)$ gives

$$\begin{aligned}
&\nabla_W \bar{G}\left(-\widehat{\nabla_W G^t_{t,t}} - \alpha_w \beta E_t \left(-\nabla_W \widehat{G^{t+1}_{t+1,t+1}}\right)\right) \\
= \; &\overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t,t}\right) + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha\beta)^k \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t,t+k} - \left(-\widehat{\nabla_W \phi}^t_{t+1,t+k}\right)\right) \\
&+ \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha\beta)^k \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t+1,t+1+k} - \left(-\widehat{\nabla_W \phi}^{t+1}_{t+1,t+1+k}\right)\right) \\
&+ E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^k (1-\alpha) \sum_{j=0}^{\infty} (\alpha_w \alpha\beta)^j \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t+k,t+k+j} - \left(-\widehat{\nabla_W \phi}^{t+1}_{t+k,t+k+j}\right)\right).
\end{aligned} \tag{150}$$

From expression (107),

$$\begin{aligned}
-\widehat{\nabla_W \phi}^t_{t,t} &= -\frac{\sigma}{1-\gamma}\hat{q}^t_t + \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \hat{n}^t + \hat{w}_t, \\
-\widehat{\nabla_W \phi}^t_{t,t+k} - \left(-\widehat{\nabla_W \phi}^t_{t+1,t+k}\right) &= -\frac{\sigma}{1-\gamma}\left(\hat{q}^t_t - \left(\hat{q}^t_{t+1} + \hat{\pi}_{t+1}\right)\right), \\
-\widehat{\nabla_W \phi}^t_{t+k,t+k+j} - \left(-\widehat{\nabla_W \phi}^{t+1}_{t+k,t+k+j}\right) &= -\frac{\sigma}{1-\gamma}\left(\hat{q}^t_{t+k} - \hat{q}^{t+1}_{t+k}\right) + \left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1}\right)\right).
\end{aligned} \tag{151}$$

Using a similar argument as above gives, using (140), (141) and (145)

$$\nabla_W \bar{G}\left(-\widehat{\nabla_W G^t_{t,t}} - \alpha_w \beta E_t \left(-\nabla_W \widehat{G^{t+1}_{t+1,t+1}}\right)\right) = R^{\Delta f,t}_{t,t} + \overline{\nabla_W \phi}\frac{1-\sigma}{1+\sigma\frac{\gamma}{1-\gamma}}\Delta\hat{n}^t, \tag{152}$$

where

$$R^{\Delta f,t}_{t,t} = \overline{\nabla_W \phi}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \hat{w}_t - \frac{\sigma}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{w}_t - \hat{w}^*_t + \frac{\gamma}{1-\gamma}\hat{x}_t\right)\right). \tag{153}$$

In terms of deviations between the sticky and flexible price equilibria, we have

$$R_{t,t}^{\Delta f,t} - R_{t,t}^{\Delta f,t*} = \overline{\nabla_W \phi} \left(\frac{1}{1-\gamma} \frac{1}{1+\sigma \frac{\gamma}{1-\gamma}} \hat{x}_t + \frac{1-\sigma}{1+\sigma \frac{\gamma}{1-\gamma}} \left(\hat{w}_t - \hat{w}_t^* \right) \right). \tag{154}$$

4.4.3 Unions $S_{t,t}^t$

Let

$$\begin{aligned}
K_1^u &= \bar{w}\bar{L}\left((1+\tau_w) - \left(\frac{\bar{V}_L}{\bar{u}_C \bar{w}} + \frac{b}{\bar{w}} \right) \right), \\
K_2^u &= \left((1+\tau_w)\, \bar{w}\bar{L} - K_1^u \sigma \frac{1}{1-\gamma} \frac{1}{1+\sigma \frac{\gamma}{1-\gamma}} \right), \\
K_3^u &= \bar{w}\bar{L}\left((1+\varepsilon_L)(1+\tau_w) - \varepsilon_L \frac{\bar{V}_L}{\bar{u}_C \bar{w}} - \varepsilon_L \frac{b}{\bar{w}} - \varepsilon_L \frac{\bar{V}_{LL}}{\bar{u}_C \bar{w}} \bar{L} \right), \qquad (155) \\
K_4^u &= \frac{\bar{w}\bar{L}}{W_t(f)} (1+\varepsilon_L)(1+\tau_w) - \sigma \frac{1}{1-\gamma} \frac{1}{W_t(f)} K_3^u \frac{1}{\left(1+\sigma \frac{\gamma}{1-\gamma}\right)}, \\
K_5^u &= \bar{L}\left(\frac{\bar{V}}{\bar{u}_C \bar{L}} - \frac{\bar{V}\left(0, \bar{Z}\right)}{\bar{u}_C \bar{L}} \right). \qquad (156)
\end{aligned}$$

Using expression (29) and (37), we have

$$\begin{aligned}
\bar{S}\hat{S}_{t,t}^t &= \bar{\Upsilon}\left(\hat{\Upsilon}_{t,t}^t - \hat{\Upsilon}_{o,t} \right) + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha \beta)^k \bar{\Upsilon}\left(\hat{\Upsilon}_{t,t+k}^t - \hat{\Upsilon}_{t+1,t+k}^t \right) \\
&+ \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \bar{\Upsilon}\left(\hat{\Upsilon}_{t+1,t+1+k}^t - \hat{\Upsilon}_{t+1,t+1+k}^{t+1} \right) \qquad (157) \\
&+ E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^{k-1} (\alpha_w - \alpha_w \alpha)\, \beta \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \bar{\Upsilon}\left(\hat{\Upsilon}_{t+k,t+k+j}^t - \hat{\Upsilon}_{t+k,t+k+j}^{t+1} \right),
\end{aligned}$$

where, using (30),

$$\begin{aligned}
\bar{\Upsilon}\hat{\Upsilon}_{t+k,t+k+j}^t &= (1+\tau_w)\, \bar{w}\bar{L}\left(\hat{n}^t - \sum_{l=1}^{k+j} \pi_{t+l}^{\omega} + \hat{w}_{t+k+j} \right) - \bar{L}\frac{\bar{V}_Z}{\bar{u}_C} \bar{Z}\hat{Z}_{t+k+j} \qquad (158) \\
&+ K_1^u \hat{L}_{t+k,t+k+j}^t (f) + \bar{L}\frac{\bar{V}}{(\bar{u}_C)^2} \left(\bar{u}_{CC}\bar{C}\hat{C}_{t+k+j} + \bar{u}_{CQ}\bar{Q}\hat{Q}_{t+k+j} \right).
\end{aligned}$$

We have, using (140), (103) and (155)

$$\bar{\Upsilon}\left(\hat{\Upsilon}_{t,t}^{t}-\hat{\Upsilon}_{o,t}\right) = K_{1}^{u}\left(-\sigma\frac{1}{1-\gamma}\hat{q}_{t}^{t}+\frac{\left(\hat{Y}_{t}-\hat{A}_{t}\right)}{1-\gamma}\right)+\left(1+\tau_{w}\right)\bar{w}\bar{L}\left(\hat{n}^{t}+\hat{w}_{t}\right)$$
$$+K_{5}^{u}\left(\bar{u}_{CC}\bar{C}\hat{C}_{t}+\bar{u}_{CQ}\bar{Q}\hat{Q}_{t}\right)v-\bar{L}\left(\frac{\bar{V}_{Z}}{\bar{u}_{C}}-\frac{\bar{V}_{Z}\left(0,\bar{Z}\right)}{\bar{u}_{C}}\right)\bar{Z}\hat{Z}_{t},$$
$$\bar{\Upsilon}\left(\hat{\Upsilon}_{t,t+k}^{t}-\hat{\Upsilon}_{t+1,t+k}^{t}\right) = -K_{1}^{u}\sigma\frac{1}{1-\gamma}\left(\hat{q}_{t}^{t}-\left(\hat{q}_{t+1}^{t}+\hat{\pi}_{t+1}\right)\right), \tag{159}$$
$$\bar{\Upsilon}\left(\hat{\Upsilon}_{t+k,t+k+j}^{t}-\hat{\Upsilon}_{t+k,t+k+j}^{t+1}\right) = -K_{1}^{u}\sigma\frac{1}{1-\gamma}\left(\hat{q}_{t+k}^{t}-\hat{q}_{t+k}^{t+1}\right)+\left(1+\tau_{w}\right)\bar{w}\bar{L}\left(\hat{n}^{t}-\hat{n}^{t+1}-\pi_{t+1}^{\omega}\right).$$

Then, using expressions (140), (141) (145), we have

$$\bar{S}\hat{S}_{t,t}^{t}=K_{2}^{u}\Delta\hat{n}^{t}+R_{t,t}^{u,t}, \tag{160}$$

where

$$R_{t,t}^{u,t} = K_{1}^{u}\frac{1}{1-\gamma}\left(\hat{Y}_{t}-\hat{A}_{t}\right)+\left(1+\tau_{w}\right)\bar{w}\bar{L}\hat{w}_{t}-\bar{L}\left(\frac{\bar{V}_{Z}}{\bar{u}_{C}}-\frac{\bar{V}_{Z}\left(0,\bar{Z}\right)}{\bar{u}_{C}}\right)\bar{Z}\hat{Z}_{t} \tag{161}$$
$$+K_{5}^{u}\left(\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_{C}}\hat{C}_{t}+\frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{C}}\hat{Q}_{t}\right)-K_{1}^{u}\frac{\frac{\sigma}{1-\gamma}}{\left(1+\sigma\frac{\gamma}{1-\gamma}\right)}\left(\hat{w}_{t}-\hat{w}_{t}^{*}+\frac{\gamma}{1-\gamma}\hat{x}_{t}\right).$$

In terms of deviations between the sticky and flexible price equilibria, we have

$$R_{t,t}^{u,t}-R_{t,t}^{u,t*} = K_{1}^{u}\frac{1}{1-\gamma}\hat{x}_{t}+\left(1+\tau_{w}\right)\bar{w}\bar{L}\left(\hat{w}_{t}-\hat{w}_{t}^{*}\right)+K_{5}^{u}\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_{C}}\hat{x}_{t} \tag{162}$$
$$-K_{1}^{u}\frac{\frac{\sigma}{1-\gamma}}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{w}_{t}-\hat{w}_{t}^{*}+\frac{\gamma}{1-\gamma}\hat{x}_{t}\right),$$

where $R_{t,t}^{u,t*}$ denotes the flexible-price version of $R_{t,t}^{u,t}$.

4.4.4 Unions $\nabla_W S^t_{t,t}$

Now consider

$$
\begin{aligned}
& \nabla_W \bar{S}\left(\widehat{\nabla_W S^t_{t,t}} - \alpha_w \beta E_t \nabla_W \widehat{S^{t+1}_{t+1,t+1}}\right) \\
= & \overline{\nabla_W \Upsilon}\widehat{\nabla_W \Upsilon}^t_{t,t} + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t,t+k} - \widehat{\nabla_W \Upsilon}^t_{t+1,t+k}\right) \\
& + \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t+1,t+1+k} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+1,t+1+k}\right) \\
& + E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^{k-1} (\alpha_w - \alpha_w \alpha)\, \beta \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \, \overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t+k,t+k+j} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+k,t+k+j}\right).
\end{aligned} \tag{163}
$$

Log-linearizing expression (42), gives

$$
\begin{aligned}
\overline{\nabla_W \Upsilon}\widehat{\nabla_W \Upsilon}^t_{t+k,t+k+j} = & \frac{1}{W_t(f)} K_3^u \hat{L}^t_{t+k,t+k+j}(f) - \varepsilon_L \frac{\bar{L}}{W_t(f)} \frac{\bar{V}_{LZ}}{\bar{u}_C} Z \hat{Z}_{t+k+j} \\
& + (1+\varepsilon_L) \frac{\bar{L}}{W_t(f)} (1+\tau_w)\, \bar{w} \left(\hat{n}^t - \sum_{l=1}^{k+j} \pi^{\omega}_{t+l} + \hat{w}_{t+k+j}\right) \\
& + \varepsilon_L \frac{\bar{w}\bar{L}}{W_t(f)} \frac{\bar{V}_L}{\bar{u}_C \bar{w}} \left(\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C} \hat{C}_{t+k+j} + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_C} \hat{Q}_{t+k+j}\right).
\end{aligned} \tag{164}
$$

Using expressions (103), (140) and (141), we can write

$$
\begin{aligned}
\overline{\nabla_W \Upsilon}\widehat{\nabla_W \Upsilon}^t_{t,t} = & \frac{\bar{L}}{W_t(f)} (1+\varepsilon_L)(1+\tau_w)\, \bar{w} \hat{n}^t \\
& + \frac{1}{W_t(f)} K_3^u \left(-\sigma \frac{1}{1-\gamma} \hat{q}^t_t\right) + J^{\Delta u,t}_{t,t}, \\
\overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t,t+k} - \widehat{\nabla_W \Upsilon}^t_{t+1,t+k}\right) = & \frac{1}{W_t(f)} K_3^u \left(-\sigma \frac{1}{1-\gamma} \left(\hat{q}^t_t - \left(\hat{q}^t_{t+1} + \hat{\pi}_{t+1}\right)\right)\right), \\
\overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t+k,t+k+j} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+k,t+k+j}\right) = & \frac{1}{W_t(f)} K_3^u \left(-\sigma \frac{1}{1-\gamma} \left(\hat{q}^t_{t+k} - \hat{q}^{t+1}_{t+k}\right)\right) \\
& + (1+\varepsilon_L) \frac{\bar{L}}{W_t(f)} (1+\tau_w)\, \bar{w} \left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1}\right)\right),
\end{aligned} \tag{165}
$$

where

$$
\begin{aligned}
J^{\Delta u,t}_{t,t} = & \frac{1}{W_t(f)} K_3^u \frac{1}{1-\gamma} \left(\hat{Y}_t - \hat{A}_t\right) + (1+\varepsilon_L) \frac{\bar{L}}{W_t(f)} (1+\tau_w)\, \bar{w} \hat{w}_t \\
& + \varepsilon_L \frac{\bar{w}\bar{L}}{W_t(f)} \frac{\bar{V}_L}{\bar{u}_C \bar{w}} \left(\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C} \hat{C}_t + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_C} \hat{Q}_t\right) - \varepsilon_L \frac{\bar{w}\bar{L}}{W_t(f)} \frac{\bar{V}_{LZ}}{\bar{u}_C \bar{w}} \bar{Z} \hat{Z}_t.
\end{aligned} \tag{166}
$$

Then we can write, using (145),

$$\nabla_W \bar{S}\left(\widehat{\nabla_W S_{t,t}^{t}} - \alpha_w \beta E_t \nabla_W \widehat{S_{t+1,t+1}^{t+1}}\right) = \frac{1}{W_t(f)} K_4^u \Delta\hat{n}^t + R_{t,t}^{\Delta u,t}, \tag{167}$$

where

$$R_{t,t}^{\Delta u,t} = J_{t,t}^{\Delta u,t} - \sigma \frac{1}{1-\gamma}\frac{1}{W_t(f)} K_3^u \frac{1}{\left(1+\sigma\frac{\gamma}{1-\gamma}\right)}\left(\hat{w}_t - \hat{w}_t^* + \frac{\gamma}{1-\gamma}\hat{x}_t\right). \tag{168}$$

In terms of differences between sticky and flexible price equilibria, we can write

$$\begin{aligned} R_{t,t}^{\Delta u,t} - R_{t,t}^{\Delta u,t*} &= \frac{1}{W_t(f)} K_3^u \frac{1}{1-\gamma}\hat{x}_t + (1+\varepsilon_L)\frac{\bar{L}}{W_t(f)}(1+\tau_w)\bar{w}(\hat{w}_t - \hat{w}_t^*) \\ &+ \varepsilon_L \frac{\bar{w}\bar{L}}{W_t(f)}\frac{\bar{V}_L}{\bar{u}_C \bar{w}}\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C}\hat{x}_t - \frac{\sigma}{1-\gamma}\frac{1}{W_t(f)} K_3^u \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{w}_t - \hat{w}_t^* + \frac{\gamma}{1-\gamma}\hat{x}_t\right), \end{aligned} \tag{169}$$

where $R_{t,t}^{\Delta u,t*}$ denotes the flexible-price version of $R_{t,t}^{\Delta u,t}$.

4.4.5 The Wage-Setting Curve

To find the wage setting equation, we first use (142) and subtract (142) in period $t+1$ multiplied by $\alpha_w \beta$. Then, we subtract the corresponding flexible-price condition. Using (146), (148), (152), (154), (160), (162), (167) and (169) gives

$$\Phi_n \Delta\hat{n}^t + \Phi_x \hat{x}_t + \Phi_w(\hat{w}_t - \hat{w}_t^*) + \Phi_n^{+1} E_t \Delta\hat{n}^{t+1} + \Phi_x^{+1} E_t \hat{x}_{t+1} + \Phi_w^{+1} E_t\left(\hat{w}_{t+1} - \hat{w}_{t+1}^*\right) = 0, \tag{170}$$

where the coefficients are

$$\begin{aligned} \Phi_n &= \varphi \frac{1}{W_t(f)} K_4^u + (1-\varphi)\frac{\nabla_W \bar{G}}{\bar{G}}\left(K_2^u + \frac{\bar{S}}{\bar{G}}\overline{tc}\right) \\ &\quad + (1-\varphi)\frac{\bar{S}}{\bar{G}}\left(-\overline{tc}\frac{1}{W_t(f)}\frac{1-\sigma}{1+\sigma\frac{\gamma}{1-\gamma}}\right), \\ \Phi_n^{+1} &= -(1-\varphi)(\alpha_w \beta)\frac{\nabla_W \bar{G}}{\bar{G}}\left(K_2^u + \frac{\bar{S}}{\bar{G}}\overline{tc}\right), \end{aligned} \tag{171}$$

$$
\begin{aligned}
\Phi_x &= \varphi\left(\varepsilon_L \frac{\bar{w}\bar{L}}{W_t(f)}\frac{\bar{V}_L}{\bar{u}_C\bar{w}}\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C} + \frac{1}{W_t(f)}K_3^u\frac{1}{1-\gamma}\frac{1}{\left(1+\sigma\frac{\gamma}{1-\gamma}\right)}\right) \\
&+ (1-\varphi)\frac{\nabla_W\bar{G}}{\bar{G}}\left(K_1^u\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}} - K_5^u\rho_C - \frac{\bar{S}}{\bar{G}}\left((1+\tau)\bar{Y} - \frac{1}{1-\gamma}\overline{tc}\right)\right) \qquad (172)\\
&+ (1-\varphi)\frac{\bar{S}}{\bar{G}}\left(-\overline{tc}\frac{1}{W_t(f)}\left(\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\right)\right), \\
\Phi_x^{+1} &= -\alpha_w\beta(1-\varphi)\frac{\nabla_W\bar{G}}{\bar{G}}\left(K_1^u\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}} - K_5^u\rho_C - \frac{\bar{S}}{\bar{G}}\left((1+\tau)\bar{Y} - \frac{1}{1-\gamma}\overline{tc}\right)\right),
\end{aligned}
$$

and, using the definitions of K_2^u and K_4^u in (155),

$$
\begin{aligned}
\Phi_n &= \Phi_w, \\
\Phi_n^{+1} &= \Phi_w^{+1}.
\end{aligned}
$$

4.4.6 With efficient taxes

From section 2.5 we have, using efficient taxes (54), that (53) holds and (173)

$$
\begin{aligned}
\bar{G} &= \left(\frac{1}{\sigma-1}+\gamma\right)\bar{Y}, \\
\nabla_W\bar{G} &= -\frac{1}{1-\alpha_w\beta}(1-\gamma)\frac{\bar{Y}}{W}, \qquad (173)\\
\bar{S} &= -\bar{w}\bar{L}\frac{\varphi\left(\frac{b}{\bar{w}} - \varepsilon_L + (1+\varepsilon_L)\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}} - \frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)\right)}{\varphi(1+\varepsilon_L) + (1-\varphi)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}.
\end{aligned}
$$

Using efficient taxes (54), we also get

$$
K_1^u = \bar{w}\bar{L}\left(\frac{-\varphi\left(\frac{b}{\bar{w}}+1\right) + (1-\varphi)\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}} - \frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}} - 1\right)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}{\varphi(1+\varepsilon_L) + (1-\varphi)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}\right), \qquad (174)
$$

$$
\begin{aligned}
K_2^u &= \bar{w}\bar{L}\frac{\varphi\varepsilon_L\left(\frac{b}{\bar{w}}+1\right) + (1-\varphi)\left(\frac{b}{\bar{w}} + \frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}} - \frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}{\varphi(1+\varepsilon_L) + (1-\varphi)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}} \\
&\quad -K_1^u\sigma\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}, \qquad (175)
\end{aligned}
$$

and

$$
\begin{aligned}
K_3^u &= \bar{w}\bar{L}\left(\frac{(1-\varphi)\left(\frac{b}{\bar{w}}-\varepsilon_L+\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}-\frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)(1+\varepsilon_L)\right)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}{\varphi(1+\varepsilon_L)+(1-\varphi)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}+\varepsilon_L\rho_L\right), \\
K_4^u &= \frac{\bar{w}\bar{L}}{W_t(f)}\left(\frac{(1-\varphi)\left(\frac{b}{\bar{w}}+(1+\varepsilon_L)\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}-\frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)-\varepsilon_L\right)\frac{1-\sigma}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}{\varphi(1+\varepsilon_L)+(1-\varphi)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}}\right) \\
&\quad +\frac{\bar{w}\bar{L}}{W_t(f)}\left(\varepsilon_L\left(1+\frac{b}{\bar{w}}-\frac{\sigma}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\rho_L\right)\right), \\
K_5^u &= \bar{L}\left(\frac{\bar{V}}{\bar{u}_C}-\frac{\bar{V}(0,\bar{Z})}{\bar{u}_C}\right).
\end{aligned} \tag{176}
$$

Moreover, setting $\overline{tc}=(1-\gamma)\bar{Y}$ in the coefficients in the wage setting equation, using (173), (174), (176) and using $\tau=\frac{1}{\sigma-1}$ gives

$$
\begin{aligned}
\Phi_n &= \varphi\frac{\bar{w}\bar{L}}{W_t(f)}\varepsilon_L\left(1+\frac{b}{\bar{w}}-\frac{\sigma}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\rho_L\right)+(1-\varphi)\frac{1}{1-\alpha_w\beta}\frac{\bar{w}\bar{L}}{W_t(f)} \\
&\quad\times\left(\frac{b}{\bar{w}}-\varepsilon_L\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}+\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}-\frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)\frac{1-\sigma}{1+\sigma\frac{\gamma}{1-\gamma}}\right)\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}, \\
\Phi_n^{+1} &= -(1-\varphi)\frac{\alpha_w\beta}{1-\alpha_w\beta}\frac{\bar{w}\bar{L}}{W_t(f)} \\
&\quad\times\left(\frac{b}{\bar{w}}-\varepsilon_L\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}+\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}-\frac{V(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)\frac{1-\sigma}{1+\sigma\frac{\gamma}{1-\gamma}}\right)\frac{(1-\sigma)}{\left(1+\frac{\gamma}{1-\gamma}\sigma\right)},
\end{aligned} \tag{177}
$$

$$
\begin{aligned}
\Phi_x &= \varphi\frac{\bar{w}\bar{L}}{W_t(f)}\varepsilon_L\left(\rho_L\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}-\rho_C\right)+(1-\varphi)\frac{1}{1-\alpha_w\beta}\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}\frac{\bar{w}\bar{L}}{W_t(f)} \\
&\quad\times\left(\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}-\frac{\bar{V}(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)\left(\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}-\rho_C\right)-\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\right), \\
\Phi_x^{+1} &= -(1-\varphi)\frac{\alpha_w\beta}{1-\alpha_w\beta}\frac{(1-\sigma)}{1+\frac{\gamma}{1-\gamma}\sigma}\frac{\bar{w}\bar{L}}{W_t(f)} \\
&\quad\times\left(\left(\frac{V(\bar{L},\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}-\frac{\bar{V}(0,\bar{Z})}{\bar{u}_C\bar{w}\bar{L}}\right)\left(\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}-\rho_C\right)-\frac{1}{1-\gamma}\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\right).
\end{aligned} \tag{178}
$$

The special case when $\varphi=1$ is interesting. We get $\frac{\Phi_n^{+1}}{\Phi_n}=\frac{\Phi_x^{+1}}{\Phi_n}=\frac{\Phi_w^{+1}}{\Phi_n}=0$ and hence we can write

$$
\Delta\hat{n}^t+\frac{\Phi_x}{\Phi_n}\hat{x}_t+(\hat{w}_t-\hat{w}_t^*)=0. \tag{179}
$$

where

$$\frac{\Phi_x}{\Phi_n} = \frac{\rho_L \frac{1}{1-\gamma} \frac{1}{1+\sigma \frac{\gamma}{1-\gamma}} - \rho_C}{1 + \frac{b}{\bar{w}} - \frac{\sigma}{1-\gamma} \frac{1}{1+\sigma \frac{\gamma}{1-\gamma}} \rho_L}. \tag{180}$$

To express the wage setting equation in terms of wage inflation, we need to express relative wages in terms of wage inflation. The wage evolution equation is, recalling that $W_t^o(f)$ is the optimal wage for firm f when renegotiating wages in period t

$$W_t = \alpha_w \int_0^1 \bar{\pi} W_{t-1}(f)\, df + (1-\alpha_w) \int_0^1 W_t^o(f)\, df. \tag{181}$$

Using that $\frac{W_{t-1}}{W_t} = \frac{1}{\pi_t^\omega}$ gives

$$1 = \alpha_w \bar{\pi} \frac{1}{\pi_t^\omega} + (1-\alpha_w) \int n^t(f)\, df. \tag{182}$$

Letting $n^t = \int n^t(f)\, df$ and log-linearizing gives

$$\hat{n}^t = \frac{\alpha_w}{1-\alpha_w} \hat{\pi}_t^\omega. \tag{183}$$

Using expression (183) in (145) yields

$$\Delta \hat{n}^t = \frac{1}{1-\alpha_w \beta} \frac{\alpha_w}{1-\alpha_w} \left(\hat{\pi}_t^\omega - \beta E_t \hat{\pi}_{t+1}^\omega \right) = \frac{1}{\Pi_1} \left(\hat{\pi}_t^\omega - \beta E_t \hat{\pi}_{t+1}^\omega \right). \tag{184}$$

Hence, letting

$$\Pi_1 = (1-\alpha_w \beta) \frac{1-\alpha_w}{\alpha_w}, \tag{185}$$

we get

$$\begin{aligned} \hat{\pi}_t^\omega - \beta E_t \hat{\pi}_{t+1}^\omega &= -\Omega_x \hat{x}_t - \Omega_w (\hat{w}_t - \hat{w}_t^*) \\ &\quad -\Omega_n^{+1} \left(E_t \hat{\pi}_{t+1}^\omega - \beta E_t \hat{\pi}_{t+2}^\omega \right) - \Omega_x^{+1} E_t \hat{x}_{t+1} - \Omega_w^{+1} E_t \left(\hat{w}_{t+1} - \hat{w}_{t+1}^* \right), \end{aligned} \tag{186}$$

where

$$\begin{aligned} \Omega_x &= \Pi_1 \frac{\Phi_x}{\Phi_n}, \\ \Omega_w &= \Pi_1 \frac{\Phi_w}{\Phi_n}, \\ \Omega_n^{+1} &= \frac{\Phi_n^{+1}}{\Phi_n}, \\ \Omega_x^{+1} &= \Pi_1 \frac{\Phi_x^{+1}}{\Phi_n}, \\ \Omega_w^{+1} &= \Pi_1 \frac{\Phi_w^{+1}}{\Phi_n}. \end{aligned} \tag{187}$$

5 Welfare

When computing welfare in this model, a second-order approximation in logs is used, resulting in that we can relate welfare to the variance in relative prices and wages. Also, the output gap matters because it distorts the economywide relationship between consumption and leisure. Before analyzing welfare, we first compute second-order approximations of L_t and Y_t, the relationship between real variation and price variation and finally persistence in price variability.

5.1 Quadratic approximation of L_t and Y_t

We first proceed by looking at a quadratic approximation of L_t and Y_t. Aggregate demand of labor by firms is, where the integral is taken over firms

$$L_t = \int_0^1 L(f)\, df. \tag{188}$$

Then a quadratic approximation is

$$\hat{L}_t = E_f \hat{L}_t(f) + \frac{1}{2} var_f \hat{L}_t(f) + o\left(\|\xi\|^3\right), \tag{189}$$

where $o\left(\|\xi\|^3\right)$ describes terms of order 3 or higher. Using the definition of the composite good in (1), we can similarly derive

$$E_f \hat{Y}_t(f) = \hat{Y}_t - \frac{1}{2}\frac{\sigma - 1}{\sigma} var_f \hat{Y}_t(f) + o\left(\|\xi\|^3\right). \tag{190}$$

Now, let us express (189) in terms of aggregate variables and variances. Taking a second-order approximation of (4) gives

$$E_f \hat{L}_t(f) = \frac{1}{1-\gamma}\left(E_f \hat{Y}_t(f) - \hat{A}_t\right) + o\left(\|\xi\|^3\right). \tag{191}$$

Then, using (190) in (191) and expression (189) we get

$$\hat{L}_t = \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) - \frac{1}{2}\frac{1}{1-\gamma}\frac{\sigma - 1}{\sigma} var_f \hat{Y}_t(f) + \frac{1}{2} var_f \hat{L}_t(f) + o\left(\|\xi\|^3\right). \tag{192}$$

5.2 Relationship between real and price variability

In this section, we relate price variability to variability in real variables, which, in turn, creates a link between price dispersion and welfare. We start by computing $var_f \hat{L}_t(f)$ as a function of $var_f \hat{P}_t(f)$ and $var_f \hat{w}_t(f)$. We also use that $w_t(f) = \frac{W_t(f)}{P_t} = n_t(f) w_t$ from (96). First, note that it follows that

$var_f \hat{w}_t(f) = var_f \hat{n}_t(f)$. Second, let us find $var_f \hat{L}_t(f)$. Since

$$\hat{L}_t(f) = -\frac{\sigma}{1-\gamma}\hat{q}_t(f) + \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + o\left(\|\xi\|^2\right), \tag{193}$$

we must have

$$var_f \hat{L}_t(f) = \left(\frac{\sigma}{1-\gamma}\right)^2 var_f \hat{q}_t(f) + o\left(\|\xi\|^3\right). \tag{194}$$

Note that, due to firm-specific labor, dispersion in $\hat{L}_t(f)$ depends directly on price dispersion, in contrast to the model in Erceg et al. (2000). Note that (194) can be rewritten as, using that $var_f \hat{q}_t(f) = var_f \hat{P}_t(f)$,

$$var_f \hat{L}_t(f) = \left(\frac{\sigma}{1-\gamma}\right)^2 var_f \hat{P}_t(f) + o\left(\|\xi\|^3\right), \tag{195}$$

and, taking a quadratic approximation of (3)

$$var_f \hat{Y}_t(f) = \sigma^2 var_f \hat{P}_t(f) + o\left(\|\xi\|^3\right). \tag{196}$$

5.3 Variance Persistence

Since prices and wages are not fully flexible, the variance of the price and wage distribution across firms are persistent. We want to find the variance of the distributions today as function of previous variances and inflation. To do this, let us express $var_f(\log P_t(f))$ and $var_f(\log W_t(f))$ in terms of squared inflation and wage inflation. Combining this with (195) and (196) we get a relationship between real variability and inflation, which enables us to write welfare in terms of inflation and wage inflation. Let $\bar{P}_t = E_f \log P_t(f)$. We have

$$var_f(\log P_t(f)) = E_f\left(\log P_t(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2, \tag{197}$$

where

$$\Delta\bar{P}_t = \bar{P}_t - \log\bar{\pi} - \bar{P}_{t-1}. \tag{198}$$

Let us rewrite $\Delta\bar{P}_t$ in terms of inflation. Since $\log P_t = E_f \log P_t(f) = \bar{P}_t$ we can rewrite $\Delta\bar{P}_t$ as[6]

$$\Delta\bar{P}_t = \hat{\pi}_t + o\left(\|\xi\|^2\right). \tag{199}$$

We also define

$$\Delta\bar{W}_t = \bar{W}_t - \log\bar{\pi} - \bar{W}_{t-1} = (1-\alpha_w)\left(\log W_t^o - \log\bar{\pi} - \bar{W}_{t-1}\right). \tag{200}$$

Similarly, let us rewrite $\Delta\bar{W}_t$ in terms of wage inflation. Note that we have $\log W_t = \bar{W}_t$.[7] Then $\Delta\bar{W}_t$ can be rewritten as

$$\Delta\bar{W}_t = \hat{\pi}_t^w + o\left(\|\xi\|^2\right). \tag{201}$$

We can write the variance in (197) as, using that when wages are changed, they are the same for all firms, i.e., $W_t^o(f) = W_t^o$ for all f,

$$\begin{aligned} & var_f(\log P_t(f)) = \alpha_w \alpha E_f\left(\log\bar{\pi}P_{t-1}(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2 \\ & + (1-\alpha)\alpha_w E_f\left(\log P_t^o(W_t(f)) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 \\ & + (1-\alpha_w)\left(\log P_t^o(W_t^o) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2. \end{aligned} \tag{202}$$

We now rewrite expression (202) in terms of lagged variance in prices, variance in wages and inflation and wage inflation. To do this, we need to rewrite the third and fourth term in expression (202). To rewrite the third term, let us express $E_f\left(\log P_t^o(W_t(f)) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2$ in terms of $\Delta\bar{P}_t$ and $\Delta\bar{W}_t$. We have

$$\Delta\bar{P}_t = (1-\alpha)\alpha_w\left(E_f\log P_t^o(W_t(f)) - \log\bar{\pi} - \bar{P}_{t-1}\right) + (1-\alpha_w)\left(E_f\log P_t^o(W_t^o) - \log\bar{\pi} - \bar{P}_{t-1}\right). \tag{203}$$

---

[6] We have

$$\begin{aligned} & \log P_t(f) - \log\bar{\pi} - \log P_{t-1}(f) \\ = & \frac{1}{\bar{P}}\left(P_t(f) - \bar{P}\right) - \log\bar{\pi} - \left(\frac{1}{\bar{P}}\left(P_{t-1}(f) - \bar{P}\right)\right) + o\left(\|\xi\|^2\right) = \hat{P}_t(f) - \log\bar{\pi} - \hat{P}_{t-1}(f) + o\left(\|\xi\|^2\right). \end{aligned}$$

Using

$$\hat{P}_t = \int_0^1 \hat{P}_t(f)\,df + o\left(\|\xi\|^2\right)$$

and integrating over f gives

$$\Delta\bar{P}_t = \hat{P}_t - \hat{P}_{t-1} - \log\bar{\pi} + o\left(\|\xi\|^2\right).$$

Since $P_t = \pi_t P_{t-1}$ we get

$$\Delta\bar{P}_t = \hat{\pi}_t + o\left(\|\xi\|^2\right).$$

[7] This follows from a similar argument as in the previous footnote.

Note that, from (5) and (10) we can write the optimal price as

$$P_t^o\left(W\left(f\right)\right) = \varrho W\left(f\right) \tag{204}$$

where ϱ only depend on aggregate variables. We thus can write

$$\log P_t^o\left(W_t^o\right) = \log P_t^o\left(\bar{\pi} W_{t-1}\left(f\right)\right) + \left(\log W_t^o - \log \bar{\pi} W_{t-1}\left(f\right)\right). \tag{205}$$

Using that we have $W_t\left(f\right) = \bar{\pi} W_{t-1}\left(f\right)$ for firms that do not change prices and since

$$E_f\left(\log \bar{\pi} W_{t-1}\left(f\right)\right) = \log \bar{\pi} + \bar{W}_{t-1} \tag{206}$$

and using (200) we have

$$\Delta \bar{P}_t - \Delta \bar{W}_t = \left(\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)\right)\left(E_f \log P_t^o\left(W_t\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right). \tag{207}$$

Also, we have

$$\begin{aligned} & E_f\left(\log P_t^o\left(W_t\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 \\ = \; & E_f\left(\log P_t^o\left(W_t\left(f\right)\right) - E_f \log P_t^o\left(W_t\left(f\right)\right) + E_f \log P_t^o\left(W_t\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 \\ = \; & var_f \log P_t^o\left(W_t\left(f\right)\right) + \left(E_f \log P_t^o\left(W_t\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2, \end{aligned} \tag{208}$$

and, using (207) and (208) we get

$$E_f\left(\log P_t^o\left(W_t\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 = var_f \log P_t^o\left(W_t\left(f\right)\right) + \frac{\left(\Delta \bar{P}_t - \Delta \bar{W}_t\right)^2}{\left(\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)\right)^2}. \tag{209}$$

To rewrite the fourth term in (202), using that $\log P_t^o\left(W_t^o\right)$ is the same for all firms that change wages and the log-linearization of $\log P_t^o\left(W_t^o\right)$ (i.e. (205)) we can write

$$\begin{aligned} & \left(\log P_t^o\left(W_t^o\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 \\ = \; & E_f\left(\log P_t^o\left(\bar{\pi} W_{t-1}\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 + E_f\left(\log W_t^o - \log \bar{\pi} W_{t-1}\left(f\right)\right)^2 \\ & +2E_f\left(\log P_t^o\left(\bar{\pi} W_{t-1}\left(f\right)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)\left(\log W_t^o - \log \bar{\pi} W_{t-1}\left(f\right)\right) + o\left(\left\|\xi\right\|^3\right). \end{aligned} \tag{210}$$

The three terms in expression (210) can be written as, using (208) and that we have $W_t\left(f\right) = \bar{\pi} W_{t-1}\left(f\right)$

for firms that do not change wages,

$$\begin{aligned}
&E_f\left(\log P_t^o\left(\bar{\pi}W_{t-1}\left(f\right)\right)-\log\bar{\pi}-\bar{P}_{t-1}\right)^2 \\
=\;& var_f \log P_t^o\left(W_t\left(f\right)\right)+\frac{1}{\left(\left(1-\alpha\right)\alpha_w+\left(1-\alpha_w\right)\right)^2}\left(\Delta\bar{P}_t-\Delta\bar{W}_t\right)^2,
\end{aligned} \tag{211}$$

using (200),

$$E_f\left(\left(\log W_t^o-\log\bar{\pi}W_{t-1}\left(f\right)\right)\right)^2=\left(\frac{1}{\left(1-\alpha_w\right)^2}\left(\Delta\bar{W}_t\right)^2+var_f\log W_{t-1}\left(f\right)\right) \tag{212}$$

and, using (200), (205) and (207),

$$\begin{aligned}
&E_f\left(\log P_t^o\left(\bar{\pi}W_{t-1}\left(f\right)\right)-\log\bar{\pi}-\bar{P}_{t-1}\right)\left(\log W_t^o-\log\bar{\pi}W_{t-1}\left(f\right)\right) \\
=\;& \frac{1}{\left(1-\alpha\right)\alpha_w+\left(1-\alpha_w\right)}\left(\Delta\bar{P}_t-\Delta\bar{W}_t\right)\frac{\Delta\bar{W}_t}{1-\alpha_w}-var_f\log W_{t-1}\left(f\right)+o\left(\left\|\xi\right\|^3\right).
\end{aligned} \tag{213}$$

Using expressions (211), (212) and (213) in (210) gives the fourth term in (202) as

$$\begin{aligned}
& var_f \log P_t^o\left(W_t\left(f\right)\right)+\frac{1}{\left(\left(1-\alpha\right)\alpha_w+\left(1-\alpha_w\right)\right)^2}\left(\Delta\bar{P}_t-\Delta\bar{W}_t\right)^2 \\
& +\left(\frac{1}{\left(1-\alpha_w\right)^2}\left(\Delta\bar{W}_t\right)^2+var_f\log W_{t-1}\left(f\right)\right) \\
& +2\left(\frac{\left(\Delta\bar{P}_t-\Delta\bar{W}_t\right)\Delta\bar{W}_t}{\left(1-\alpha\right)\alpha_w+\left(1-\alpha_w\right)}\frac{1}{1-\alpha_w}-var_f\log W_{t-1}\left(f\right)\right)+o\left(\left\|\xi\right\|^3\right).
\end{aligned} \tag{214}$$

Let us now collect the arguments above to rewrite expression (202) in terms of lagged variance in prices, variance in wages and inflation and wage inflation. The expression $var_f \log P_t^o\left(W_t\left(f\right)\right)$ involves firms that do not change wages. From (204) we then have[8]

$$var_f\log P_t^o\left(W_t\left(f\right)\right)=var_f\log W_t\left(f\right)=var_f\log W_{t-1}\left(f\right)+o\left(\left\|\xi\right\|^3\right). \tag{215}$$

Then we have, using (209), (214) and (215) in (202),

$$\begin{aligned}
var_f\left(\log P_t\left(f\right)\right) \;=\;& \alpha_w\alpha var_f\left(\log P_{t-1}\left(f\right)\right)+\left(1-\alpha\right)\alpha_w var_f\log W_{t-1}\left(f\right) \\
& +\frac{\alpha_w}{\left(1-\alpha\right)\alpha_w+\left(1-\alpha_w\right)}\left(\alpha\left(\Delta\bar{P}_t\right)^2+\frac{1-\alpha}{1-\alpha_w}\left(\Delta\bar{W}_t\right)^2\right)+o\left(\left\|\xi\right\|^3\right).
\end{aligned} \tag{216}$$

[8]Noting that the variance $var_f \log W_t\left(f\right)$ is computed over firms that do not change wages, implying $var_f\log W_t\left(f\right)=var_f\log W_{t-1}\left(f\right)$.

Using expressions (199) and (201) gives

$$\begin{aligned} var_f\left(\log P_t\left(f\right)\right) &= \alpha_w \alpha var_f\left(\log P_{t-1}\left(f\right)\right) + \left(1-\alpha\right)\alpha_w var_f \log W_{t-1}\left(f\right) \\ &+ \frac{\alpha_w}{\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)}\left(\alpha\left(\hat{\pi}_t\right)^2 + \frac{1-\alpha}{1-\alpha_w}\left(\hat{\pi}_t^w\right)^2\right) + o\left(\|\xi\|^3\right). \end{aligned} \tag{217}$$

For wages, we can write, using a similar method as in (197), and using (200) we have

$$var_f\left(\log W_t\left(f\right)\right) = \alpha_w var_f\left(\log W_{t-1}\left(f\right)\right) + \frac{\alpha_w}{1-\alpha_w}\left(\Delta \bar{W}_t\right)^2. \tag{218}$$

Using expression (201) this gives

$$var_f\left(\log W_t\left(f\right)\right) = \alpha_w var_f\left(\log W_{t-1}\left(f\right)\right) + \frac{\alpha_w}{1-\alpha_w}\left(\hat{\pi}_t^w\right)^2 + o\left(\|\xi\|^3\right). \tag{219}$$

5.4 Welfare

When analyzing the welfare in the model, we focus on the limiting cashless economy. The social welfare function is then

$$\sum_{t=0}^{\infty}\beta^t SW_t, \tag{220}$$

where

$$SW_t = u\left(C_t, Q_t\right) - \int_0^1 V\left(L_t\left(f\right), Z_t\right)df. \tag{221}$$

Taking a second-order approximation of $u\left(C_t, Q_t\right)$ gives

$$\begin{aligned} u\left(C_t, Q_t\right) &= \bar{u} + \bar{u}_C\bar{C}\left(\hat{C}_t + \frac{1}{2}\left(\hat{C}_t\right)^2\right) + \bar{u}_Q\bar{Q}\left(\hat{Q}_t + \frac{1}{2}\left(\hat{Q}_t\right)^2\right) + \frac{1}{2}\bar{u}_{CC}\bar{C}^2\left(\hat{C}_t\right)^2 \\ &+ \bar{u}_{CQ}\bar{C}\bar{Q}\hat{C}_t\hat{Q}_t + \frac{1}{2}\bar{u}_{QQ}\bar{Q}^2\left(\hat{Q}_t\right)^2 + o\left(\|\xi\|^3\right). \end{aligned} \tag{222}$$

Let us take a second order approximation of $V\left(L_t\left(f\right), Z_t\right)$,using the standard variance decomposition $E_f\left(\hat{L}_t\right)^2 = var_f\hat{L}_t + \left(E_f\hat{L}_t\right)^2$. Using (192) to eliminate $E_f\hat{L}_t\left(f\right)$ and $\left(E_f\hat{L}_t\right)^2$ gives

$$\begin{aligned} & E_f V\left(L_t\left(f\right)\right) = \bar{V}_L\bar{L}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) - \frac{1}{2}\frac{1}{1-\gamma}\frac{\sigma-1}{\sigma}var_f\hat{Y}_t\left(f\right) + \frac{1}{2}var_f\hat{L}_t\left(f\right)\right) \\ & + \left(\bar{V}_L\bar{L} + \bar{V}_{LL}\bar{L}^2\right)\frac{1}{2}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)^2 + \frac{1}{2}\bar{V}_{LL}\bar{L}^2 var_f\hat{L}_t\left(f\right) \\ & + \bar{V}_{LZ}\bar{L}\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\bar{Z}\hat{Z}_t + tip + o\left(\|\xi\|^3\right) \end{aligned} \tag{223}$$

where *tip* denotes terms that are independent of policy. Since $\hat{Z}_t$ is and aggregate (and thus common) disturbance we have $E_f\hat{Z}_t = \hat{Z}_t$.[9]

Combining the second order approximations of $u(C_t, Q_t)$ and $E_f V(L_t(f), Z_t)$ from expressions (222) and (223), gives welfare as, using $\hat{C}_t = \hat{Y}_t$,[10]

$$\begin{aligned} SW_t &= \frac{1}{2}\left(\bar{u}_C\bar{C} + \bar{u}_{CC}\bar{C}^2\right)\left(\hat{Y}_t\right)^2 + \bar{u}_{CQ}\bar{C}\bar{Q}\hat{Y}_t\hat{Q}_t - \frac{\bar{V}_L\bar{L}}{2}\left(var_f\hat{L}_t(f) - \frac{1}{1-\gamma}\frac{\sigma-1}{\sigma}var_f\hat{Y}_t(f)\right) \\ &\quad -\frac{1}{2}\bar{V}_{LL}\bar{L}^2 var_f\hat{L}_t(f) - \left(\bar{V}_L\bar{L} + \bar{V}_{LL}\bar{L}^2\right)\frac{1}{2}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)^2 \\ &\quad -\bar{V}_{LZ}\bar{L}\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\bar{Z}\hat{Z}_t + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{224}$$

We are interested in computing the difference between sticky and flexible-price welfare. Consider welfare when prices are flexible. Note that there is no variance in the price and wage distribution across firms, since all prices and wages are adjusted in every period. Let us analyze the difference $SW_t - SW_t^*$, i.e. the welfare difference,

$$\begin{aligned} SW_t - SW_t^* &= \frac{1}{2}\bar{u}_C\bar{C}\left(-\rho_C + \rho_L\frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma}\right)\left(\left(\hat{Y}_t\right)^2 - \left(\hat{Y}_t^*\right)^2\right) - \frac{1}{2}\bar{V}_{LL}\bar{L}^2 var_f\hat{L}_t(f) \\ &\quad + \left(\bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t - \frac{\bar{V}_{LZ}\bar{L}}{1-\gamma}\bar{Z}\hat{Z}_t + \frac{\left(\bar{V}_L\bar{L} + \bar{V}_{LL}\bar{L}^2\right)}{(1-\gamma)^2}\hat{A}_t\right)\left(\hat{Y}_t - \hat{Y}_t^*\right) \\ &\quad -\bar{V}_L\bar{L}\left(-\frac{1}{2}\frac{1}{1-\gamma}\frac{\sigma-1}{\sigma}var_f\hat{Y}_t(f) + \frac{1}{2}var_f\hat{L}_t(f)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{225}$$

Let us eliminate the shock terms by using that flexible-price output $\tilde{Y}_t^*$ is a function of the disturbances in the model. Using expression (71) in expression (225) gives, where we use the definitions of ρ_C and that $\bar{u}_C\overline{MPL} = \bar{V}_L$,

$$\begin{aligned} SW_t - SW_t^* &= -\Lambda^*\bar{C}\hat{Y}_t^*\left(\hat{Y}_t - \hat{Y}_t^*\right) + \frac{\Lambda^*\bar{C}}{2}\left(\left(\hat{Y}_t\right)^2 - \left(\hat{Y}_t^*\right)^2\right) - \frac{1}{2}\bar{V}_{LL}\bar{L}^2 var_f\hat{L}_t(f) \\ &\quad -\bar{V}_L\bar{L}\left(-\frac{1}{2}\frac{1}{1-\gamma}\frac{\sigma-1}{\sigma}var_f\hat{Y}_t(f) + \frac{1}{2}var_f\hat{L}_t(f)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{226}$$

Note that the first row on the right hand side can be rewritten as $\frac{\Lambda\bar{C}}{2}\left(\hat{Y}_t - \hat{Y}_t^*\right)^2$. Using (**??**), (195),

[9] Note that the terms $\left(var_f\hat{Y}_t\right)^2$, $var_f\hat{Y}_t var_f\hat{L}_t(f)$, $\left(var_f\hat{L}_t(f)\right)^2$, $\left(\hat{Y}_t - \hat{A}_t\right)var_f\hat{Y}_t$ and $\left(\hat{Y}_t - \hat{A}_t\right)var_f\hat{L}_t(f)$ appearing in the $\left(E_f\hat{L}_t\right)^2$ term vanishes since they are of order three or higher.

[10] The terms involving only the disturbances are independent of policy.

(196) and (226), the total welfare difference is

$$
\begin{aligned}
E_0 \sum_{t=0}^{\infty} \beta^t \left(SW_t - SW_t^*\right) &= E_0 \frac{\Lambda^* \bar{C}}{2} \sum_{t=0}^{\infty} \beta^t \left(\hat{Y}_t - \hat{Y}_t^*\right)^2 \\
&\quad - E_0 \bar{u}_C \left(1-\gamma\right) \bar{Y} \frac{1}{2} \left(-\frac{1}{1-\gamma} \frac{\sigma - 1}{\sigma} \sigma^2 + \left(\frac{\sigma}{1-\gamma}\right)^2 \right) \sum_{t=0}^{\infty} \beta^t var_f \hat{P}_t\left(f\right) \\
&\quad - E_0 \frac{1}{2} \bar{u}_C \left(1-\gamma\right) \bar{Y} \frac{\bar{V}_{LL} \bar{L}}{\bar{V}_L} \left(\frac{\sigma}{1-\gamma}\right)^2 \sum_{t=0}^{\infty} \beta^t var_f \hat{P}_t\left(f\right) + tip + o\left(\left\|\xi\right\|^3\right).
\end{aligned}
\tag{227}
$$

Since there is a direct relationship between relative price and wage variability, as indicated by (204), there is no unique solution for the objective. To highlight the differences between our model and the model in Erceg et al. (2000) we rewrite our the model so that only the coefficient in front of wage inflation parameter differs from the model in Erceg et al. (2000). We thus get

$$
\begin{aligned}
SW_t - SW_t^* &= \frac{\Lambda^* \bar{C}}{2} \left(\hat{Y}_t - \hat{Y}_t^*\right)^2 \\
&\quad - \bar{u}_C \left(1-\gamma\right) \bar{Y} \frac{1}{2} \left(\frac{\sigma}{1-\gamma} var_f \hat{P}_t\left(f\right) + \gamma \left(\frac{\sigma}{1-\gamma}\right)^2 var_f \hat{w}_t\left(f\right) \right) \\
&\quad - \frac{1}{2} \bar{u}_C \left(1-\gamma\right) \bar{Y} \frac{\bar{V}_{LL} \bar{L}}{\bar{V}_L} \left(\frac{\sigma}{1-\gamma}\right)^2 var_f \hat{w}_t\left(f\right) + tip + o\left(\left\|\xi\right\|^3\right).
\end{aligned}
\tag{228}
$$

This gives the same coefficient on the inflation variability term as in Erceg et al. (2000). Repeatedly substituting (219) into itself (forwardly), using (199), starting at period 0 gives

$$
var_f\left(\log W_t\left(f\right)\right) = \left(\alpha_w\right)^{t+1} var_f\left(\log W_{-1}\left(f\right)\right) + \frac{\alpha_w}{1-\alpha_w} \sum_{s=0}^{t} \alpha_w^{t-s} \left(\hat{\pi}_s^w\right)^2 + o\left(\left\|\xi\right\|^3\right). \tag{229}
$$

Multiplying by β^t on both sides, using that $var_f\left(\log W_{-1}\left(f\right)\right)$ is independent of policy and summing from period 0 to infinity gives [11]

$$
E_0 \sum_{t=0}^{\infty} \beta^t var_f\left(\log W_t\left(f\right)\right) = \frac{\alpha_w}{1-\alpha_w} \frac{1}{1-\alpha_w \beta} E_0 \sum_{t=0}^{\infty} \beta^t \left(\hat{\pi}_t^w\right)^2 + tip + o\left(\left\|\xi\right\|^3\right). \tag{230}
$$

[11] We use the following rearrangement of the double sum

$$
\sum_{t=0}^{\infty} \sum_{s=0}^{t} \left(\alpha_w \beta\right)^t \left(\alpha_w\right)^{-s} \left(\hat{\pi}_s^w\right)^2 = \sum_{s=0}^{\infty} \sum_{t=s}^{\infty} \left(\alpha_w \beta\right)^t \left(\alpha_w\right)^{-s} \left(\hat{\pi}_s^w\right)^2.
$$

Now consider price variability again. Expression (217) can be rewritten as

$$
\begin{aligned}
var_f\left(\log P_t(f)\right) &= \alpha_w \alpha var_f\left(\log P_{t-1}(f)\right)+(1-\alpha)\,\alpha_w var_f \log W_{t-1}(f) \\
&+\frac{\alpha_w\alpha}{(1-\alpha)\,\alpha_w+(1-\alpha_w)}\left(\hat{\pi}_t\right)^2+\frac{(1-\alpha)\,\alpha_w}{(1-\alpha_w)\left((1-\alpha)\,\alpha_w+(1-\alpha_w)\right)}\left(\hat{\pi}_t^w\right)^2 .
\end{aligned} \tag{231}
$$

Repeatedly substituting (231) into itself (forwardly), starting at period 0 and taking expectations at period 0 gives

$$
\begin{aligned}
E_0 var_f\left(\log P_t(f)\right) &= E_0\sum_{s=0}^{t-1}\left(\alpha_w\alpha\right)^{t-1-s}\left((1-\alpha)\,\alpha_w\right)var_f\left(\log W_s(f)\right) \\
&+E_0\sum_{s=0}^{t}\left(\alpha_w\alpha\right)^{t-s}\frac{(1-\alpha)\,\alpha_w}{(1-\alpha_w)\left((1-\alpha)\,\alpha_w+(1-\alpha_w)\right)}\left(\hat{\pi}_s^w\right)^2 \\
&+E_0\sum_{s=0}^{t}\left(\alpha_w\alpha\right)^{t-s}\frac{\alpha_w\alpha}{(1-\alpha)\,\alpha_w+(1-\alpha_w)}\left(\hat{\pi}_s\right)^2+tip+o\left(\|\xi\|^3\right).
\end{aligned} \tag{232}
$$

Multiplying by β^t on both sides and summing from period 0 to infinity gives [12]

$$
\begin{aligned}
E_0\sum_{t=0}^{\infty}\beta^t var_f\left(\log P_t(f)\right) &= \beta\frac{(1-\alpha)\,\alpha_w}{1-\beta\alpha_w\alpha}E_0\sum_{t=0}^{\infty}\beta^t var_f\left(\log W_t(f)\right) \\
&+\frac{(1-\alpha)\,\alpha_w}{(1-\beta\alpha_w\alpha)(1-\alpha_w)\left((1-\alpha)\,\alpha_w+(1-\alpha_w)\right)}E_0\sum_{t=0}^{\infty}\beta^t\left(\hat{\pi}_t^w\right)^2 \\
&+\frac{\alpha_w\alpha}{(1-\beta\alpha_w\alpha)\left((1-\alpha)\,\alpha_w+(1-\alpha_w)\right)}E_0\sum_{t=0}^{\infty}\beta^t\left(\hat{\pi}_t\right)^2+tip+o\left(\|\xi\|^3\right).
\end{aligned} \tag{233}
$$

$$
E_0\sum_{t=0}^{\infty}\beta^t var_f\left(\log W_t(f)\right)=\frac{\alpha_w}{1-\alpha_w}\frac{1}{1-\alpha_w\beta}E_0\sum_{t=0}^{\infty}\beta^t\left(\hat{\pi}_t^w\right)^2+tip+o\left(\|\xi\|^3\right). \tag{234}
$$

Now we are able to state welfare in terms of squared inflation, wage inflation and output gap. From (227), (230) and (233) we get

$$
E_0\sum_{t=0}^{\infty}\beta^t\left(SW_t-SW_t^*\right)=E_0\sum_{t=0}^{\infty}\beta^t\mathcal{L}_t+tip+o\left(\|\xi\|^3\right), \tag{235}
$$

[12] We rewrite the double sum

$$
\begin{aligned}
\sum_{t=0}^{\infty}\beta^t\sum_{s=0}^{t}\left(\alpha_w\alpha\right)^{t-s} &= \sum_{s=0}^{\infty}\sum_{t=s}^{\infty}\beta^t\left(\alpha_w\alpha\right)^{t-s}=\sum_{s=0}^{\infty}\left(\alpha_w\alpha\right)^{-s}\sum_{t=s}^{\infty}\beta^t\left(\alpha_w\alpha\right)^t=\sum_{s=0}^{\infty}\frac{\beta^s}{1-\beta\alpha_w\alpha}, \\
\sum_{t=0}^{\infty}\beta^t\sum_{s=0}^{t-1}\left(\alpha_w\alpha\right)^{t-1-s} &= \sum_{s=0}^{\infty}\sum_{t=s+1}^{\infty}\beta^t\left(\alpha_w\alpha\right)^{t-1-s}=\sum_{s=0}^{\infty}\sum_{r=s}^{\infty}\beta^{r+1}\left(\alpha_w\alpha\right)^{r-s}=\beta\sum_{s=0}^{\infty}\sum_{r=s}^{\infty}\beta^r\left(\alpha_w\alpha\right)^{r-s}.
\end{aligned}
$$

where

$$\mathcal{L}_t = \theta_x (\hat{x}_t)^2 + \theta_\pi (\hat{\pi}_t)^2 + \theta_{\pi^\omega} (\hat{\pi}_t^\omega)^2, \tag{236}$$

and, defining

$$\Pi_W = \frac{1 - \alpha_w \alpha}{\alpha_w \alpha} (1 - \alpha_w \alpha \beta)$$

and using definition (185) we have

$$\begin{aligned}
\theta_x &= \frac{\Lambda^* \bar{C}}{2} = \frac{\bar{u}_C \bar{C}}{2} \left(-\rho_C + \rho_L \frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma} \right), && (237) \\
\theta_\pi &= -\left(\bar{u}_C \bar{C} \frac{1}{2} \sigma^2 \left(\frac{1-\sigma}{\sigma} + \frac{1-\rho_L}{1-\gamma} \right) \right) \frac{1}{\Pi_W}, && (238) \\
\theta_{\pi^\omega} &= -\left(\bar{u}_C \bar{C} \frac{1}{2} \sigma^2 \left(\frac{1-\sigma}{\sigma} + \frac{1-\rho_L}{1-\gamma} \right) \right) \left(-\frac{1}{\Pi_W} + \frac{1}{\Pi_1} \right). && (239)
\end{aligned}$$

6 Policy

In this section, we solve the model, both if policy follows a simple rule and if monetary policy is optimal.

6.1 A Simple Rule

We assume that the central bank follows the rule

$$\hat{I}_t = \rho_I \hat{I}_{t-1} + (1 - \rho_I) (\gamma_\pi \hat{\pi}_t + \gamma_x \hat{x}_t). \tag{240}$$

We could also introduce a monetary policy shock. However, here we are primarily interested in technology shocks, since we will compare the outcome under a simple rule versus the outcome under optimal policy.

We need to rewrite the shock process (78) and the system of constraints (113), (130), (186) (85) on the form in Söderlind (1999). To do this, we define the auxiliary variables

$$\hat{\varpi}_t = \hat{w}_t - (\hat{\pi}_t^\omega - \hat{\pi}_t) \tag{241}$$

and

$$\hat{c}_t = \Omega_n^{+1} \left(\hat{\pi}_t^\omega - \beta E_t \hat{\pi}_{t+1}^\omega \right) + \Omega_x^{+1} \hat{x}_t + \Omega_w^{+1} (\hat{w}_t - \hat{w}_t^*). \tag{242}$$

Also, using the policy rule (240) and Phillips curve (130) to rewrite the Euler equation gives

$$
\begin{aligned}
E_t\hat{x}_{t+1} &= \frac{\rho_I}{\rho_C}\hat{I}_{t-1} + \frac{(1-\rho_I)}{\rho_C}\left(\gamma_\pi\hat{\pi}_t + \gamma_x\hat{x}_t\right) + (1-\eta)\frac{1-\rho_L}{\rho_C-\rho_L}\hat{w}_t^* - \frac{1}{\rho_C\beta}\hat{\pi}_t \\
&+ \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{1}{\rho_C\beta\Omega_n^{+1}}\hat{c}_t + \frac{1}{\rho_C\beta}\left(\Pi - \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{\Omega_w^{+1}}{\Omega_n^{+1}}\right)\left(\hat{\varpi}_t + (\hat{\pi}_t^\omega - \hat{\pi}_t) - \hat{w}_t^*\right) \qquad (243) \\
&+ \frac{1}{\rho_C\beta}\left(\frac{\gamma}{1-\gamma}\Pi - \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{\Omega_x^{+1}}{\Omega_n^{+1}}\right)\hat{x}_t + \hat{x}_t.
\end{aligned}
$$

Defining $x_t = (x_{1t}, x_{2t})$ where

$$
\begin{aligned}
x_{1t} &= \left(\hat{w}_t^*, \hat{\varpi}_t, \hat{I}_{t-1}\right)', \qquad (244) \\
x_{2t} &= \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^\omega, \hat{c}_t\right)'
\end{aligned}
$$

we can write

$$
E_t x_{t+1} = A x_t + \epsilon_t, \qquad (245)
$$

where

$$
\epsilon_t' = \begin{pmatrix} \varepsilon_t & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (246)
$$

Defining

$$
\begin{aligned}
K_1^S &= \Pi - \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{\Omega_w^{+1}}{\Omega_n^{+1}}, \\
K_2^S &= \Omega_w - \frac{\Omega_w^{+1}}{\Omega_n^{+1}}, \\
a_{41} &= (1-\eta)\frac{1-\rho_L}{\rho_C-\rho_L} - \frac{1}{\rho_C\beta}K_1^S, \qquad (247) \\
a_{44} &= \frac{(1-\rho_I)}{\rho_C}\gamma_x + \frac{1}{\rho_C\beta}\left(\frac{\gamma}{1-\gamma}\Pi - \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{\Omega_x^{+1}}{\Omega_n^{+1}}\right) + 1, \\
a_{45} &= \frac{(1-\rho_I)}{\rho_C}\gamma_\pi - \frac{1}{\rho_C\beta} - \frac{1}{\rho_C\beta}K_1^S,
\end{aligned}
$$

the matrix A can be written as

$$A = \begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & \rho_I & (1-\rho_I)\,\gamma_x & (1-\rho_I)\,\gamma_\pi & 0 & 0 \\ a_{41} & \frac{K_1^S}{\rho_C \beta} & \frac{\rho_I}{\rho_C} & a_{44} & a_{45} & \frac{K_1^S}{\rho_C \beta} & \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}} \frac{1}{\rho_C \beta \Omega_n^{+1}} \\ \frac{K_1^S}{\beta} & -\frac{K_1^S}{\beta} & 0 & -\frac{1}{\beta}\left(\frac{\gamma}{1-\gamma}\Pi - \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{\Omega_x^{+1}}{\Omega_n^{+1}}\right) & \frac{K_1^S}{\beta} + \frac{1}{\beta} & -\frac{K_1^S}{\beta} & -\frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{1}{\beta\Omega_n^{+1}} \\ -\frac{\Omega_w^{+1}}{\Omega_n^{+1}\beta} & \frac{\Omega_w^{+1}}{\Omega_n^{+1}\beta} & 0 & \frac{\Omega_x^{+1}}{\Omega_n^{+1}\beta} & -\frac{\Omega_w^{+1}}{\Omega_n^{+1}\beta} & \frac{\Omega_n^{+1}+\Omega_w^{+1}}{\Omega_n^{+1}\beta} & -\frac{1}{\Omega_n^{+1}\beta} \\ K_2^S & -K_2^S & 0 & \frac{\Omega_x^{+1}}{\Omega_n^{+1}} - \Omega_x & K_2^S & -K_2^S & -\frac{1}{\Omega_n^{+1}} \end{pmatrix}. \tag{248}$$

6.2 Optimal Policy

Again, we need to rewrite the shock process (78) and the system of constraints (113), (130), (186) and the shock process on the form used in Söderlind (1999). The system of constraints is

$$\begin{aligned} \hat{w}_t^* &= \rho \hat{w}_{t-1}^* + \varepsilon_t, \\ \hat{w}_t &= \hat{w}_{t-1} + \hat{\pi}_t^\omega - \hat{\pi}_t, \\ \hat{\pi}_t &= \beta E_t \hat{\pi}_{t+1} + \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\left(\hat{\pi}_t^\omega - \beta E_t \hat{\pi}_{t+1}^\omega\right) + \Pi\left(\hat{w}_t - \hat{w}_t^*\right) + \frac{\gamma}{1-\gamma}\Pi \hat{x}_t, \\ \hat{\pi}_t^\omega &= \beta E_t \hat{\pi}_{t+1}^\omega - \Omega_x \hat{x}_t - \Omega_w\left(\hat{w}_t - \hat{w}_t^*\right) \\ &\quad -\Omega_n^{+1}\left(E_t \hat{\pi}_{t+1}^\omega - \beta E_t \hat{\pi}_{t+2}^\omega\right) - \Omega_x^{+1} E_t \hat{x}_{t+1} - \Omega_w^{+1} E_t\left(\hat{w}_{t+1} - \hat{w}_{t+1}^*\right). \end{aligned} \tag{249}$$

Defining

$$\begin{aligned} x_t &= \left(\hat{w}_t^*, \hat{\varpi}_t, \hat{\pi}_t, \hat{\pi}_t^\omega, \hat{c}_t\right)', \\ u_t &= \hat{x}_t \end{aligned} \tag{250}$$

and

$$\epsilon_t' = \begin{pmatrix} \varepsilon_t & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{251}$$

and partitioning as in Söderlind (1999) gives

$$E_t x_{t+1} = A x_t + B u_t + \xi_{t+1}, \tag{252}$$

and defining

$$K_\pi^1 = \left(\Pi - \frac{1}{1+\sigma\frac{\gamma}{1-\gamma}}\frac{\Omega_w^{+1}}{\Omega_n^{+1}}\right) \tag{253}$$

and

$$K_A^{op} = \frac{1}{\beta\Omega_n}\left(\Omega_w - \Pi\Omega_n - \gamma\Omega_w\right), \tag{254}$$

we have

$$\begin{aligned}
A &= \begin{pmatrix} \rho & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ \frac{1}{\beta}K_\pi & -\frac{1}{\beta}K_\pi & \frac{1}{\beta}\left(K_\pi+1\right) & -\frac{1}{\beta}K_\pi & \frac{\gamma-1}{\beta\Omega_n-\beta\gamma\Omega_n+\sigma\beta\gamma\Omega_n} \\ \frac{1}{\beta\Omega_n}\Omega_w & -\frac{1}{\beta\Omega_n}\Omega_w & \frac{1}{\beta\Omega_n}\Omega_w & -\frac{1}{\beta\Omega_n}\left(\Omega_n+\Omega_w\right) & \frac{1}{\beta\Omega_n} \\ -\frac{1}{\Omega_n}\Omega_w\left(\Omega_n-1\right) & \frac{1}{\Omega_n}\Omega_w\left(\Omega_n-1\right) & -\frac{1}{\Omega_n}\Omega_w\left(\Omega_n-1\right) & \frac{1}{\Omega_n}\Omega_w\left(\Omega_n-1\right) & \frac{1}{\Omega_n} \end{pmatrix}, \\
B &= \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{\beta\Omega_n(\gamma-1)(\sigma\gamma-\gamma+1)}\left(\Omega_x - 2\gamma\Omega_x + \gamma^2\Omega_x + \gamma^2\Pi\Omega_n - \gamma\Pi\Omega_n - \sigma\gamma^2\Pi\Omega_n\right) \\ \frac{1}{\beta\Omega_n}\Omega_x \\ -\frac{1}{\Omega_n}\Omega_x\left(\Omega_n-1\right) \end{pmatrix}, \\
\xi_t &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}\varepsilon_t.
\end{aligned} \tag{255}$$

Given the definition of x_t we can write Q, U and R as

$$\begin{aligned}
Q &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_\pi & 0 & 0 \\ 0 & 0 & 0 & -\theta_{\pi^\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
U &= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
R &= -\theta_x.
\end{aligned} \tag{256}$$

7 The Erceg et al. (2000) model

Since the sticky price equilibrium is derived as in Erceg et al. (2000), we do not reproduce the derivations here. Condition (113) is identical in the two models. The conditions corresponding to (130) and (186) are, with firm specific capital,

$$\begin{aligned} \hat{\pi}_t &= \beta E_t \hat{\pi}_{t+1} + \Pi\left(\hat{w}_t - \hat{w}_t^*\right) + \frac{\gamma}{1-\gamma}\Pi \hat{x}_t, \\ \hat{\pi}_t^{\omega} &= \beta E_t \hat{\pi}_{t+1}^{\omega} - \Omega_w^E\left(\hat{w}_t - \hat{w}_t^*\right) + \Omega_x^E \hat{x}_t, \end{aligned} \tag{257}$$

where Π and Π_1 are defined as in (129) and (185), respectively, and

$$\begin{aligned} \Omega_w^E &= \frac{\Pi_1}{1-\rho_L \sigma_w}, \\ \Omega_x^E &= \Omega_w^E\left(\rho_C - \rho_L \frac{1}{1-\gamma}\right). \end{aligned} \tag{258}$$

With freely mobile capital, the Phillips curve is

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \Pi_W\left(\hat{w}_t - \hat{w}_t^*\right) + \frac{\gamma}{1-\gamma}\Pi_W \hat{x}_t$$

and the wage setting curve is identical to (257).

7.0.1 The EHL model with families.

Labor demand for labor j by firm f is

$$N_{t+k}^t(f,j) = \left(\frac{\bar{\pi}^k W_t(j)}{W_{t+k}}\right)^{-\sigma_w} N_{t+k}(f) \tag{259}$$

where $N_{t+k}(f)$ is total labor demand by firm f and

$$\frac{\bar{\pi}^k W_t(j)}{W_{t+k}} = n_t(f)\, X_{t,k}^{\omega} \tag{260}$$

Loglinearizing gives

$$\hat{N}_{t+k}^t(f,j) = -\sigma_w\left(\hat{n}^t - \sum_{l=1}^{k+j} \pi_{t+l}^{\omega}\right) + \hat{N}_{t+k}(f) \tag{261}$$

Wages are determined by

$$\nabla_W S_{t,t}^t = 0 \tag{262}$$

where $\nabla_W S_{t,t}^t$ is as in (41) with $\nabla_W \Upsilon_{t+k+j}^t$ replaced by $\nabla_W \Upsilon_{t+k,t+k+j}^t$ and

$$\begin{aligned}\nabla_W \Upsilon_{t+k+j}^t(j) &= \varepsilon_L \frac{N_{t+k+j}^t(j)}{W_t(j)}\left((1+\tau_w)\frac{(\bar{\pi})^{k+j} W_t(j)}{P_{t+k+j}} - b\right) \\ &\quad -\varepsilon_L \frac{\left(N_{t+k+j}^t(j)\right)}{W_t(j)} \frac{V_N\left(N_{t+k+j}^t(j), Z_t\right)}{u_{C,t+k+j}} + \frac{N_{t+k+j}^t(j)}{W_t(f)}(1+\tau_w)\frac{\bar{\pi}^{k+j} W_t(j)}{P_{t+k+j}}.\end{aligned} \tag{263}$$

where $\hat{N}_{t+k+j}^t(j)$ is the loglinearization of the employment variable

$$\varepsilon_L = -\sigma_w \tag{264}$$

Loglinearizing gives

$$\begin{aligned}\overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}_{t+k+j}^t &= \frac{1}{W_t(f)} K_3^u \hat{N}_{t+k+j}^t(j) - \varepsilon_L \frac{\bar{N}}{W_t(f)} \frac{\bar{V}_{NZ}}{\bar{u}_C} Z \hat{Z}_{t+k+j} \\ &\quad + (1+\varepsilon_L)\frac{\bar{N}}{W_t(f)}(1+\tau_w)\bar{w}\left(\hat{n}^t - \sum_{l=1}^{k+j}\pi_{t+l}^{\omega} + \hat{w}_{t+k+j}\right) \\ &\quad + \varepsilon_L \frac{\bar{w}\bar{N}}{W_t(f)} \frac{\bar{V}_N}{\bar{u}_C \bar{w}}\left(\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C}\hat{C}_{t+k+j} + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_C}\hat{Q}_{t+k+j}\right).\end{aligned} \tag{265}$$

Employment in a firm is

$$\left(N_{t+k}(f)\right)^{\frac{\sigma_w - 1}{\sigma_w}} = \int_0^1 \left(N_{t+k}^t(f,j)\right)^{\frac{\sigma_w - 1}{\sigma_w}} dj \tag{266}$$

Loglinearizing gives

$$\hat{N}_{t+k}(f) = \int_0^1 \hat{N}_{t+k}^t(f,j)\, dj \tag{267}$$

The technology is

$$Y_t(f) = A_t N_t(f)^{1-\gamma} K_t(f)^{\gamma} \tag{268}$$

and hence

$$\hat{Y}_t(f) = \hat{A}_t + (1-\gamma)\hat{N}_t(f) \tag{269}$$

Integrating over all firms gives

$$\hat{N}_t = \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) \tag{270}$$

Integrating labor demand over firms

$$\hat{N}_{t+k}^t(j) = -\sigma_w\left(\hat{n}^t - \sum_{l=1}^{k+j}\pi_{t+l}^{\omega}\right) + \hat{N}_{t+k} = -\sigma_w\left(\hat{n}^t - \sum_{l=1}^{k+j}\pi_{t+l}^{\omega}\right) + \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) \tag{271}$$

Taxes are, using $\varphi = 1$

$$1 + \tau_w = \frac{\varepsilon_L}{(1+\varepsilon_L)} \left(\frac{b}{\bar{w}} + 1 \right). \tag{272}$$

and hence, using $\bar{V}_N = \bar{u}_C \bar{w}$ and letting $\rho_L = -\frac{\bar{V}_{NN}\bar{N}}{\bar{V}_N}$

$$\begin{aligned} & \nabla_W \bar{S} \left(\widehat{\nabla_W S^t_{t,t}} - \alpha_w \beta E_t \nabla_W \widehat{S^{t+1}_{t+1,t+1}} \right) \\ = \; & \overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}^t_t + \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \beta)^k \, \overline{\nabla_W \Upsilon} \left(\widehat{\nabla_W \Upsilon}^t_{t+1+k} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+1+k} \right) \end{aligned} \tag{273}$$

where

$$\begin{aligned} \nabla_W \Upsilon^t_{t+k,t+k+j} \;=\; & \varepsilon_L \frac{L^t_{t+k,t+k+j}(f)}{W_t(f)} \left((1+\tau_w) \frac{(\bar{\pi})^{k+j} W_t(f)}{P_{t+k+j}} - b \right) \\ & - \varepsilon_L \frac{L^t_{t+k,t+k+j}(f)}{W_t(f)} \frac{V_L \left(L^t_{t+k,t+k+j}(f), Z_{t+k+j} \right)}{u_{C,t+k+j}} \\ & + \frac{L^t_{t+k,t+k+j}(f)}{W_t(f)} (1+\tau_w) \frac{\bar{\pi}^{k+j} W_t(f)}{P_{t+k+j}}, \end{aligned} \tag{274}$$

$$\begin{aligned} \overline{\nabla_W \Upsilon} \left(\widehat{\nabla_W \Upsilon}^t_{t+k} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+k} \right) \;=\; & \frac{1}{W_t(f)} K^u_3 \left(\hat{N}^t_{t+k}(j) - \hat{N}^{t+1}_{t+k}(j) \right) \\ & + (1+\varepsilon_L) \frac{\bar{N}}{W_t(f)} (1+\tau_w) \bar{w} \left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1} \right) \right) \end{aligned} \tag{275}$$

and

$$\left(\hat{N}^t_{t+k}(j) - \hat{N}^{t+1}_{t+k}(j) \right) = -\sigma_w \left(\hat{n}^t - \hat{n}^{t+1} - \pi^{\omega}_{t+1} \right). \tag{276}$$

Hence

$$\begin{aligned} \overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}^t_t \;=\; & \frac{1}{W_t(f)} K^u_3 \hat{N}^t_t(j) + \frac{\bar{N}}{W_t(f)} \varepsilon_L \left(\frac{b}{\bar{w}} + 1 \right) \bar{w} \left(\hat{n}^t + \hat{w}_t \right) \\ & + \varepsilon_L \frac{\bar{w}\bar{N}}{W_t(f)} \frac{\bar{V}_N}{\bar{u}_C \bar{w}} \left(\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C} \hat{C}_{t+k+j} + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_C} \hat{Q}_{t+k+j} \right) \\ \overline{\nabla_W \Upsilon} \left(\widehat{\nabla_W \Upsilon}^t_{t+k} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+k} \right) \;=\; & -\frac{1}{W_t(f)} K^u_3 \sigma_w \left(\hat{n}^t - \hat{n}^{t+1} - \pi^{\omega}_{t+1} \right) \\ & + \frac{\bar{L}}{W_t(f)} \varepsilon_L \left(1 + \frac{b}{\bar{w}} \right) \bar{w} \left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1} \right) \right) \end{aligned} \tag{277}$$

where

$$\begin{aligned} K_3^u &= \frac{wL}{W_t\left(f\right)}\varepsilon_L\rho_L, \\ K_4^u &= \frac{wL}{W_t\left(f\right)}\varepsilon_L\left(\frac{b}{\bar{w}}+1+\varepsilon_L\rho_L\right). \end{aligned} \tag{278}$$

Repeating a similar argument as in section 4.4 gives

$$\nabla_W\bar{S}\left(\widehat{\nabla_W S_{t,t}^t}-\alpha_w\beta E_t\nabla_W\widehat{S_{t+1,t+1}^{t+1}}\right)=\frac{1}{W_t\left(f\right)}K_4^u\Delta\hat{n}^t+R_{t,t}^{\Delta u,t} \tag{279}$$

where

$$R_{t,t}^{\Delta u,t}-R_{t,t}^{\Delta u,t*}=\frac{1}{W_t\left(f\right)}\left(K_3^u\frac{1}{1-\gamma}\hat{x}_t+\bar{w}\bar{N}\varepsilon_L\left(\frac{b}{\bar{w}}+1\right)\left(\hat{w}_t-\hat{w}_t^*\right)+\varepsilon_L\bar{w}\bar{N}\frac{\bar{V}_N}{\bar{u}_C\bar{w}}\frac{\bar{u}_{CC}\bar{C}}{\bar{u}_C}\hat{x}_t\right) \tag{280}$$

We then get $\Phi_n^{+1}=\Phi_x^{+1}=\Phi_w^{+1}=0$ and

$$\begin{aligned} \Phi_n &= \frac{wL}{W_t\left(f\right)}\varepsilon_L\left(\frac{b}{\bar{w}}+1+\varepsilon_L\rho_L\right) \\ \Phi_x &= \frac{\bar{w}\bar{N}}{W_t\left(f\right)}\varepsilon_L\left(\rho_L\frac{1}{1-\gamma}-\rho_C\right) \\ \Phi_w &= \frac{1}{W_t\left(f\right)}\bar{w}\bar{N}\varepsilon_L\left(\frac{b}{\bar{w}}+1\right) \end{aligned} \tag{281}$$

and hence, in terms of the model above in (257), we have

$$\begin{aligned} \Omega_w^E &= \Pi_1\frac{\Phi_w}{\Phi_n}=\Pi_1\frac{\frac{b}{\bar{w}}+1}{\frac{b}{\bar{w}}+1+\varepsilon_L\rho_L}, \\ \Omega_x^E &= -\Pi_1\frac{\Phi_x}{\Phi_n}=-\Pi_1\frac{\rho_L\frac{1}{1-\gamma}-\rho_C}{\frac{b}{\bar{w}}+1+\varepsilon_L\rho_L}. \end{aligned} \tag{282}$$

If $b=0$ as in Erceg et al. (2000), the expression can be simplified further to

$$\begin{aligned} \Omega_w^E &= \Pi_1\frac{1}{1-\sigma_w\rho_L}, \\ \Omega_x^E &= \Pi_1\frac{\rho_C-\rho_L\frac{1}{1-\gamma}}{1-\sigma_w\rho_L}. \end{aligned} \tag{283}$$

7.1 Welfare

When computing welfare in this model, a second-order approximation in logs is used, resulting in that we can relate welfare to the variance in relative prices and wages. Also, the output gap matters because it distorts the economywide relationship between consumption and leisure. Here, we compute the loss function parameters in a model with freely mobile labor. Derivations of coefficients with firm-specific capital can be found in Galí (2008). Before analyzing welfare, we first compute second-order approximations of L_t and Y_t, the relationship between real variation and price variation and finally persistence in price variability.

7.2 Quadratic approximation of L_t and Y_t

We first proceed by looking at a quadratic approximation of L_t and Y_t. Aggregate demand of labor by firms is, where the integral is taken over firms

$$L_t = \int_0^1 L(f)\, df. \tag{284}$$

Then a quadratic approximation is

$$\hat{L}_t = E_f \hat{L}_t(f) + \frac{1}{2} var_f \hat{L}_t(f) + o\left(\|\xi\|^3\right). \tag{285}$$

Using the definition of the composite good in (1), we can similarly derive

$$E_f \hat{Y}_t(f) = \hat{Y}_t - \frac{1}{2}\frac{\sigma - 1}{\sigma} var_f \hat{Y}_t(f) + o\left(\|\xi\|^3\right). \tag{286}$$

Now, let us express (285) in terms of aggregate variables and variances. Composite labor in Erceg et al. (2000) is given by

$$L_t = \left(\int_0^1 N_t(j)^{\frac{\sigma_w - 1}{\sigma_w}} dj\right)^{\frac{\sigma_w}{\sigma_w - 1}}. \tag{287}$$

By a similar argument to (286), we get

$$E_j \hat{N}_t(j) = \hat{L}_t - \frac{1}{2}\frac{\sigma_w - 1}{\sigma_w} var_j \hat{N}_t(j) + o\left(\|\xi\|^3\right). \tag{288}$$

As in expression B.11 in Erceg et al. (2000), we have

$$E_f \hat{Y}_t(f) = \hat{A}_t - \gamma \hat{L}_t + E_f \hat{L}_t(f) + o\left(\|\xi\|^3\right) \tag{289}$$

and, noting that capital labor ratios are the same for all firms, we have

$$var_f \hat{Y}_t(f) = var_f \hat{L}_t(f) + o\left(\|\xi\|^3\right), \tag{290}$$

since there is no local variation in $\hat{A}_t$. Using (285), (286) and (290) gives

$$\hat{L}_t = \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \frac{1}{2}\frac{1}{\sigma(1-\gamma)} var_f \hat{Y}_t(f) + o\left(\|\xi\|^3\right). \tag{291}$$

Then, using (291), (288) can be rewritten as

$$E_j \hat{N}_t(j) = \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \frac{1}{2}\frac{1}{\sigma(1-\gamma)} var_f \hat{Y}_t(f) - \frac{1}{2}\frac{\sigma_w - 1}{\sigma_w} var_j \hat{N}_t(j) + o\left(\|\xi\|^3\right). \tag{292}$$

7.3 Relationship between real and price variability

Using a quadratic approximation of (3)

$$var_f \hat{Y}_t(f) = \sigma^2 var_f \hat{P}_t(f) + o\left(\|\xi\|^3\right) \tag{293}$$

and similarly for labor demand, derived from (287)

$$var_j \hat{N}_t(j) = \sigma_w^2 var_j \hat{w}_t(j) + o\left(\|\xi\|^3\right). \tag{294}$$

7.4 Variance Persistence

Since prices and wages are not fully flexible, the variance of the price and wage distribution across firms are persistent. We want to find the variance of the distributions today as function of previous variances and inflation. To do this, let us express $var_f(\log P_t(f))$ and $var_j(\log W_t(j))$ in terms of squared inflation and wage inflation. Combining this with (294) and (293) we get a relationship between real variability and inflation, which enables us to write welfare in terms of inflation and wage inflation. Let $\bar{P}_t = E_f \log P_t(f)$. We have, using expression (199)

$$var_f(\log P_t(f)) = E_f\left(\log P_t(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2. \tag{295}$$

We can write the variance in (295) as

$$\begin{aligned} var_f(\log P_t(f)) = {} & \alpha_w \alpha E_f\left(\log\bar{\pi} P_{t-1}(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2 \\ & + (1-\alpha_w\alpha)\left(\log P_t^o - \log\bar{\pi} - \bar{P}_{t-1}\right)^2. \end{aligned} \tag{296}$$

We now rewrite expression (296) in terms of lagged variance in prices and inflation. To do this, we need to rewrite the second and third term in expression (296) in terms of inflation. First, note that we have

$$\Delta\bar{P}_t = (1-\alpha_w\alpha)\left(\log P_t^o - \log\bar{\pi} - \bar{P}_{t-1}\right). \tag{297}$$

Then we have, using (199) and the expression above in (296)

$$var_f\left(\log P_t\left(f\right)\right) = \alpha_w \alpha var_f\left(\log P_{t-1}\left(f\right)\right) + \frac{\alpha_w \alpha}{1-\alpha_w \alpha}\left(\hat{\pi}_t\right)^2 + o\left(\|\xi\|^3\right). \tag{298}$$

For wages, we can write, using a similar method as when deriving (298)

$$var_j\left(\log W_t\left(j\right)\right) = \alpha_w var_j\left(\log W_{t-1}\left(j\right)\right) + \frac{\alpha_w}{1-\alpha_w}\left(\hat{\pi}_t^w\right)^2 + o\left(\|\xi\|^3\right). \tag{299}$$

7.5 Welfare

When analyzing the welfare in the model, we focus on the limiting cashless economy. The social welfare function is then

$$\sum_{t=0}^{\infty} \beta^t SW_t, \tag{300}$$

with SW_t defined as

$$SW_t = u\left(C_t, Q_t\right) - \int_0^1 V\left(N_t\left(j\right), Z_t\right) dj. \tag{301}$$

Taking a second-order approximation of $u\left(C_t, Q_t\right)$ is identical to our model; see expression (222).

Log-linearizing the second term in (301) gives, using the standard variance decomposition $E_j\left(\hat{N}_t\right)^2 = var_j \hat{N}_t\left(j\right) + \left(E_j \hat{N}_t\right)^2$ and expression (292) for $\hat{N}_t$. Since $\hat{Z}_t$ is aggregate we have $E_j \hat{Z}_t = \hat{Z}_t$ and hence[13]

$$\begin{aligned} E_j V\left(N_t\left(j\right), Z_t\right) = {} & \bar{V}_N \bar{N}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \frac{1}{2}\frac{1}{\sigma\left(1-\gamma\right)} var_f \hat{Y}_t\left(f\right) - \frac{1}{2}\frac{\sigma_w - 1}{\sigma_w} var_j \hat{N}_t\left(j\right)\right) \\ & + \bar{V}_N \bar{N}\left(\frac{1}{2}\left(var_j \hat{N}_t\left(j\right) + \left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t - \hat{A}_t\right)^2\right)\right) + \bar{V}_Z \bar{Z}\left(\hat{Z}_t + \frac{1}{2}\left(\hat{Z}_t\right)^2\right) \\ & + \frac{1}{2}\bar{V}_{NN}\bar{N}^2\left(var_j \hat{N}_t\left(j\right) + \left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t - \hat{A}_t\right)^2\right) \\ & + \bar{V}_{NZ}\bar{N}\bar{Z}\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\hat{Z}_t + \frac{1}{2}\bar{V}_{ZZ}\bar{Z}^2\left(\hat{Z}_t\right)^2 + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{302}$$

Combining the log linearizations of $u\left(C_t^j, Q_t\right)$ and $\int_0^1 V\left(N_t\left(j\right), Z_t\right) dj$ from expressions (222) and

[13] Note that the terms $\left(var_f \hat{Y}_t\left(f\right)\right)^2$, $var_f \hat{Y}_t\left(f\right) var_j \hat{N}_t\left(j\right)$, $\left(var_j \hat{N}_t\left(j\right)\right)^2$, $\left(\hat{Y}_t - \hat{A}_t\right)\left(var_f \hat{Y}_t\left(f\right)\right)$ and $\left(\hat{Y}_t - \hat{A}_t\right)\left(var_j \hat{N}_t\left(j\right)\right)$ appearing in the $\left(E_j \hat{N}_t\left(j\right)\right)^2$ term vanish since they are of order 3 and higher.

(302) gives welfare as

$$
\begin{aligned}
SW_t &= \bar{u}_C\bar{C}\left(\hat{C}_t+\frac{1}{2}\left(\hat{C}_t\right)^2\right)+\frac{1}{2}\bar{u}_{CC}\bar{C}^2\left(\hat{C}_t\right)^2+\bar{u}_{CQ}\bar{C}\bar{Q}\hat{C}_t\hat{Q}_t \qquad (303)\\
&-\bar{V}_N\bar{N}\left(\frac{\hat{Y}_t-\hat{A}_t}{1-\gamma}+\frac{1}{2}\frac{1}{\sigma(1-\gamma)}var_f\hat{Y}_t(f)+\frac{1}{2}\frac{1}{\sigma_w}var_j\hat{N}_t(j)+\frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t-\hat{A}_t\right)^2\right)\\
&-\frac{1}{2}\bar{V}_{NN}\bar{N}^2\left(var_j\hat{N}_t(j)+\left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t-\hat{A}_t\right)^2\right)-\bar{V}_{NZ}\bar{N}\bar{Z}\frac{1}{1-\gamma}\left(\hat{Y}_t-\hat{A}_t\right)\hat{Z}_t\\
&+tip+o\left(\|\xi\|^3\right). \qquad (304)
\end{aligned}
$$

We are interested in computing the difference between sticky and flexible price welfare. The difference $SW_t - SW_t^*$ is, using that $\bar{u}_C\bar{C}(1-\gamma)=\bar{V}_N\bar{N}$,

$$
\begin{aligned}
SW_t-SW_t^* &= \left(\bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t+\frac{\bar{V}_N\bar{N}+\bar{V}_{NN}\bar{N}^2}{(1-\gamma)^2}\hat{A}_t-\frac{\bar{V}_{NZ}\bar{N}\bar{Z}}{1-\gamma}\hat{Z}_t\right)\left(\hat{Y}_t-\hat{Y}_t^*\right)\\
&+\frac{1}{2}\left(\bar{u}_C\bar{C}-\left(\frac{1}{1-\gamma}\right)^2\bar{V}_N\bar{N}+\bar{u}_{CC}\bar{C}^2-\bar{V}_{NN}\bar{N}^2\right)\left(\left(\hat{Y}_t\right)^2-\left(\hat{Y}_t^*\right)^2\right) \qquad (305)\\
&-\frac{\bar{V}_N\bar{N}}{2}\left(\frac{1}{\sigma(1-\gamma)}var_f\hat{Y}_t(f)+\frac{1}{\sigma_w}var_j\hat{N}_t(j)\right)-\frac{\bar{V}_{NN}\bar{N}^2}{2}var_j\hat{N}_t(j)\\
&+tip+o\left(\|\xi\|^3\right). \qquad (306)
\end{aligned}
$$

We can eliminate the shock terms by using that flexible price output $\tilde{Y}_t^*$ can be written as a function of shocks. As in our model, see expression (71), we can write

$$
\Lambda^*\bar{C}\hat{Y}_t^*=-\bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t+\frac{\bar{Z}\bar{N}}{(1-\gamma)}\bar{V}_{NZ}\hat{Z}_t-\frac{\bar{N}}{1-\gamma}\left(\bar{V}_N+\bar{V}_{NN}\bar{N}\right)\frac{1}{1-\gamma}\hat{A}_t, \qquad (307)
$$

where

$$
\Lambda^*=\bar{u}_{CC}\bar{C}^*-\bar{V}_{NN}\frac{\bar{N}}{\overline{\overline{MPL}}}\frac{1}{1-\gamma}-\frac{1}{\overline{\overline{MPL}}}\bar{V}_N\frac{\gamma}{1-\gamma}. \qquad (308)
$$

Using the expression above for $\Lambda^*\bar{C}\hat{Y}_t^*$ in expression (305) for $SW_t - SW_t^*$ gives

$$
\begin{aligned}
SW_t-SW_t^* &= \frac{\Lambda^*\bar{C}}{2}\left(\hat{Y}_t-\hat{Y}_t^*\right)^2-\frac{\bar{V}_{NN}\bar{N}^2}{2}var_j\hat{N}_t(j) \qquad (309)\\
&-\bar{v}_N\bar{N}\frac{1}{2}\left(\frac{1}{\sigma(1-\gamma)}var_f\hat{Y}_t(f)+\frac{1}{\sigma_w}var_j\hat{N}_t(j)\right)+tip+o\left(\|\xi\|^3\right).
\end{aligned}
$$

Using (309), and (294) and (293), the total welfare difference is

$$
var_j\hat{N}_t(j)=\sigma_w^2 var_j\hat{w}_t(j)
$$

$$\sum_{t=0}^{\infty}\beta^t\left(SW_t - SW_t^*\right) = \frac{\Lambda\bar{C}}{2}\sum_{t=0}^{\infty}\beta^t\left(\hat{Y}_t - \hat{Y}_t^*\right)^2 - \sum_{t=0}^{\infty}\beta^t\frac{\bar{V}_{NN}\bar{N}^2}{2}\sigma_w^2 var_j\hat{w}_t(j) - \bar{V}_N\bar{N}\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(\frac{\sigma}{1-\gamma}var_f\hat{P}_t(f) + \sigma_w var_j\hat{W}_t(j)\right) + tip + o\left(\|\xi\|^3\right). \tag{310}$$

Repeatedly substituting expression (298) into itself (forwardly), starting at 0 gives

$$var_f\left(\log P_t(f)\right) = \left(\alpha_w\alpha\right)^{t+1}var_f\left(\log P_{-1}(f)\right) + \frac{\alpha_w\alpha}{1-\alpha_w\alpha}\sum_{s=0}^{t}\left(\alpha_w\alpha\right)^{t-s}\left(\hat{\pi}_s\right)^2 + o\left(\|\xi\|^3\right). \tag{311}$$

Multiplying by β^t on both sides and summing from 0 to infinity gives

$$\sum_{t=0}^{\infty}\beta^t var_f\left(\log P_t(f)\right) = \frac{\alpha_w\alpha}{1-\alpha_w\alpha}\frac{1}{1-\alpha_w\alpha\beta}\sum_{t=0}^{\infty}(\beta)^t\left(\hat{\pi}_t\right)^2 + tip + o\left(\|\xi\|^3\right). \tag{312}$$

The same can be done for $var_j\left(\log W_t(j)\right)$. We get

$$\sum_{t=0}^{\infty}\beta^t var_f\left(\log W_t(f)\right) = \frac{\alpha_w}{1-\alpha_w}\frac{1}{1-\alpha_w\beta}\sum_{t=0}^{\infty}(\beta)^t\left(\hat{\pi}_t^{\omega}\right)^2 + tip + o\left(\|\xi\|^3\right). \tag{313}$$

Using expressions (312) and (313) in (310) gives

$$\sum_{t=0}^{\infty}\beta^t\left(SW_t - SW_t^*\right) = \sum_{t=0}^{\infty}\beta^t L_t + tip + o\left(\|\xi\|^3\right), \tag{314}$$

where

$$L_t = \theta_x\left(\hat{x}_t\right)^2 + \theta_\pi\left(\hat{\pi}_t\right)^2 + \theta_{\pi^\omega}\left(\hat{\pi}_t^{\omega}\right)^2 \tag{315}$$

and

$$\begin{aligned}\theta_x &= \frac{\Lambda^*\bar{C}}{2} = \frac{\bar{u}_C\bar{C}}{2}\left(-\rho_C + \rho_L\frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma}\right),\\ \theta_\pi &= -\frac{1}{2}\bar{u}_C\bar{C}\sigma\frac{1}{\Pi_W},\\ \theta_\omega &= -\frac{1}{2}\bar{u}_C\bar{C}\left(1-\gamma\right)\sigma_w\left(1-\rho_L\sigma_w\right)\frac{1}{\Pi_1}.\end{aligned} \tag{316}$$

With firm-specific capital, the loss function coefficients for the output gap and wage inflation are as above, while the inflation coefficient is

$$\theta_\pi = -\frac{1}{2}\bar{u}_C\bar{C}\sigma\frac{1}{\Pi}.$$

7.6 A Simple Rule

Similar arguments as in section 6.1 establishes that the Erceg et al. (2000) can be written as $E_t x_{t+1} = Ax_t + \epsilon_t$ where

$$x_t = \left(\hat{w}_t^*, \hat{\varpi}_t, \hat{I}_{t-1}, \hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^{\omega}\right)' \tag{317}$$

and

$$A = \begin{pmatrix} \eta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & \rho_I & (1-\rho_I)\gamma_x & (1-\rho_I)\gamma_\pi & 0 \\ a_{41} & \frac{1}{\rho_C}\frac{1}{\beta}\Pi & \frac{\rho_I}{\rho_C} & a_{44} & a_{45} & \frac{1}{\rho_C}\frac{1}{\beta}\Pi \\ \frac{\Pi}{\beta} & -\frac{\Pi}{\beta} & 0 & -\frac{1}{\beta}\frac{\gamma}{1-\gamma}\Pi & \frac{\Pi}{\beta}+\frac{1}{\beta} & -\frac{\Pi}{\beta} \\ -\frac{1}{\beta}\Omega_w^E & \frac{1}{\beta}\Omega_w^E & 0 & -\frac{1}{\beta}\Omega_x^E & -\frac{1}{\beta}\Omega_w^E & \frac{1}{\beta}\Omega_w^E + \frac{1}{\beta} \end{pmatrix} \tag{318}$$

where

$$\begin{aligned} a_{41} &= (1-\eta)\frac{1-\rho_L}{\rho_C-\rho_L} - \frac{1}{\rho_C}\frac{1}{\beta}\Pi, \\ a_{44} &= \frac{1-\rho_I}{\rho_C}\gamma_x + 1 + \frac{1}{\rho_C}\frac{1}{\beta}\frac{\gamma}{1-\gamma}\Pi, \\ a_{45} &= \frac{1-\rho_I}{\rho_C}\gamma_\pi - \frac{1}{\rho_C}\frac{1}{\beta} - \frac{1}{\rho_C}\frac{1}{\beta}\Pi. \end{aligned} \tag{319}$$

7.7 Optimal Policy

To solve for optimal policy (with firm-specific capital), the central bank maximizes (314), subject to

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \Pi(\hat{w}_t - \hat{w}_t^*) + \frac{\gamma}{1-\gamma}\Pi\hat{x}_t, \tag{320}$$

$$\hat{w}_t = \hat{w}_{t-1} + \hat{\pi}_t^{\omega} - \hat{\pi}_t, \tag{321}$$

$$\hat{\pi}_t^{\omega} = \beta E_t \hat{\pi}_{t+1}^{\omega} - \Omega_w^E(\hat{w}_t - \hat{w}_t^*) + \Omega_x^E \hat{x}_t, \tag{322}$$

$$\hat{w}_{t+1}^* = \rho\hat{w}_t^* + \varepsilon_{t+1}. \tag{323}$$

We follow the method in Söderlind (1999) to solve for optimal policy under commitment and discretion. As above, we rewrite the system in terms of $\hat{\varpi}_t$ as defined in (241). We get

$$\hat{w}_{t+1}^* = \rho\hat{w}_t^* + \varepsilon_{t+1}, \tag{324}$$

$$\hat{\varpi}_{t+1} = \hat{\varpi}_t + (\hat{\pi}_t^{\omega} - \hat{\pi}_t), \tag{325}$$

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{\gamma}{1-\gamma}\Pi\hat{x}_t + \Pi(\hat{\varpi}_t + (\hat{\pi}_t^{\omega} - \hat{\pi}_t) - \hat{w}_t^*), \tag{326}$$

$$\hat{\pi}_t^{\omega} = \beta E_t \hat{\pi}_{t+1}^{\omega} - \Omega_w^E(\hat{\varpi}_t + (\hat{\pi}_t^{\omega} - \hat{\pi}_t) - \hat{w}_t^*) + \Omega_x^E \hat{x}_t. \tag{327}$$

Defining

$$x_t = \left(\hat{w}_t^*, \hat{\varpi}_t, \hat{\pi}_t, \hat{\pi}_t^\omega\right)', \tag{328}$$

we have, in terms of the notation in Söderlind (1999),

$$A = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ \frac{1}{\beta}\Pi & -\frac{1}{\beta}\Pi & \frac{1}{\beta}\left(\Pi+1\right) & -\frac{1}{\beta}\Pi \\ -\frac{1}{\beta}\Omega_w^E & \frac{1}{\beta}\Omega_w^E & -\frac{1}{\beta}\Omega_w^E & \frac{1}{\beta}\left(\Omega_w^E+1\right) \end{pmatrix} \tag{329}$$

and

$$B = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{\beta}\gamma\frac{\Pi}{\gamma-1} \\ -\frac{1}{\beta}\Omega_x^E \end{pmatrix} \tag{330}$$

where

$$\begin{aligned} \Pi &= \frac{1-\alpha}{\alpha}\left(1-\alpha\beta\right), \\ \Pi_1 &= \left(1-\alpha_w\beta\right)\frac{1-\alpha_w}{\alpha_w}, \\ \Omega_w^E &= \frac{\Pi_1}{1-\rho_L\sigma_w}, \\ \Omega_x^E &= \Omega_w^E\left(\rho_C-\rho_L\frac{1}{1-\gamma}\right). \end{aligned} \tag{331}$$

Also,

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta_\pi & 0 \\ 0 & 0 & 0 & -\theta_\omega \end{pmatrix}, \tag{332}$$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \tag{333}$$

and $R = -\theta_x$.

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