Technical Appendix to "Monetary Policy and Staggered Wage Bargaining when Prices are Sticky"*

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Abstract

In this paper we describe in detail how to derive the model used in the paper "Monetary Policy and Staggered Wage Bargaining when Prices are Sticky" by Mikael Carlsson and Andreas Westermark. We present the model, loglinearize it and proceed to solve for optimal discretionary monetary policy.

Keywords: Monetary Policy, Bargaining, DSGE Models.

JEL classification: E52, E58, J41.

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1 Introduction

In this paper the model in Carlsson and Westermark (2006) is presented in detail. We describe the agents and sectors of the economy and state the conditions for optimizing behavior. We then describe how to compute the steady state of the model. Next, we proceed to loglinearize the flexible as well as the sticky price model around the steady state. We first loglinearize the optimal price- and wage-setting decisions, and then the wage flow equation. We then proceed to derive a second-order log approximation of the welfare function of the sticky price model. Finally, we solve for the optimal discretionary monetary policy.

In section 2, we outline the model. In section 3 and 4 we loglinearize the flexible and sticky price models, respectively, and in section 5 the log quadratic approximation of welfare is derived. Finally, in section 6 we solve for optimal policy.

2 The Economic Environment

There is a competitive final goods sector with flexible prices and a monopolistically competitive intermediate goods sector where producers set prices in staggered contracts as in Calvo (1983). To each firm a household is attached. Thus, in contrast to Erceg, Henderson, and Levin (2000), firms do not perceive workers as atomistic. In each period, wages are renegotiated with a fixed probability. Thus, wages are staggered as in Calvo (1983) but, in contrast to Erceg, Henderson, and Levin (2000), they are determined in bargaining between the union and the firm and not unilaterally by the union.

2.1 Final goods firms

Since we assume complete contingent claims markets (except for leisure), households are identical, except for leisure choices, it then simplifies the analysis to abstract away from the households optimal choices for individual goods. We follow Erceg, Henderson, and Levin (2000) and assume a competitive sector selling a composite final good. The composite good is combined from individual or intermediate goods in the same proportions that households would choose. The composite good is

$$Y_t = \left[\int_0^1 Y_t\left(f\right)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{1}$$

where $\sigma > 1$ and $Y_t(f)$ is the intermediate good produced by intermediate goods firm f. The price P_t of one unit of the composite good is set equal to marginal cost

$$P_t = \left[\int_0^1 P_t(f)^{1-\sigma} df \right]^{\frac{1}{1-\sigma}}. \tag{2}$$

2.2 Intermediate good firms

By standard arguments, the demand function for the generic good f from the final goods sector is

$$Y_{t+k}(f) = \left(\frac{\bar{\pi}^k P_t(f)}{P_{t+k}} \right)^{-\sigma} Y_{t+k}. \tag{3}$$

Intermediate goods firms produce according to the following constant returns production function

$$Y_t(f) = A_t K_t(f)^{\gamma} L_t(f)^{1-\gamma}, \tag{4}$$

where A_t is the technology level, common to all firms, and $K_t(f)$ and $L_t(f)$ denote the firms capital and labor input in period t, respectively. Since firms have the right to manage, $K_t(f)$ and $L_t(f)$ are chosen optimally, taking the rental cost of capital and the wage $W_t(f)$ as given. Moreover, as in e.g. Erceg, Henderson, and Levin (2000), the aggregate capital stock is fixed at $\bar{K}$. Solving for capital and labor choices in the cost minimization problem gives, letting $\Gamma = \gamma^{-\gamma}(1-\gamma)^{-(1-\gamma)}$

$$\begin{aligned} K_t(f) &= \gamma\Gamma \frac{Y_t(f)}{A_t} (W_t(f))^{1-\gamma} (P_t^c)^{\gamma-1}, \\ L_t(f) &= (1-\gamma)\Gamma \frac{Y_t(f)}{A_t} (W_t(f))^{-\gamma} (P_t^c)^{\gamma}. \end{aligned} \tag{5}$$

The cost and marginal cost functions for firm f are then given by

$$\begin{aligned} TC(W_t(f), Y_t(f)) &= \Gamma \frac{Y_t(f)}{A_t} (W_t(f))^{1-\gamma} (P_t^c)^{\gamma}, \\ MC(W_t(f), Y_t(f)) &= \Gamma \frac{1}{A_t} (W_t(f))^{1-\gamma} (P_t^c)^{\gamma}, \end{aligned} \tag{6}$$

respectively. The marginal product is in real terms, ignoring the time period when the contract was signed

$$MPL_t(f) = (1-\gamma) A_t \left(\frac{w_t(f)}{p_t^c} \right)^{\gamma} \left(\frac{1-\gamma}{\gamma} \right)^{-\gamma}, \tag{7}$$

where $w_t(f) = \frac{W_t(f)}{P_t}$ and $p_t^c = \frac{P_t^c}{P_t}$ are the real wages and real capital prices, respectively.[1] Furthermore, real costs is given by

$$tc\left(w_t(f), p_t^c, Y_t(f)\right) = \Gamma \frac{Y_t(f)}{A_t} \left(w_t(f)\right)^{1-\gamma} \left(p_t^c\right)^{\gamma}. \tag{8}$$

2.3 Calvo price and wage determination with indexation

Firms are allowed to change prices in a given period with probability $1-\alpha$ and to renegotiate wages with probability $1-\alpha_w$. Any firm that renegotiates wages, is also allowed to change prices. The probability that prices are unchanged is $\alpha_w \alpha$. This assumption simplifies our problem greatly, since it eliminates any intertemporal interdependence in price-setting decisions for a given firm. We assume that prices are indexed by the steady-state inflation rate, as in Yun (1996).

2.3.1 Prices

The producers choose prices to maximize

$$\max_{P_t(f)} E_t \sum_{k=0}^{\infty} \left(\alpha_w \alpha\right)^k \Psi_{t,t+k} \left[(1+\tau)\,\bar{\pi}^k P_t(f)\, Y_{t+k}(f) - TC\left(W_t(f), Y_t(f)\right)\right] \tag{9}$$

$$\text{s. t. } \; Y_{t+k}(f) = \left(\frac{\bar{\pi}^k P_t(f)}{P_{t+k}}\right)^{-\sigma} Y_{t+k}.$$

Note that the term within the square brackets is just the firm's profit in period $t+k$, given that prices were last reset in period t. The term $\Psi_{t,t+k}$ captures households valuation of nominal profits in period $t+k$. This will in general depend on time preferences β^k and the marginal utility in period $t+k$. The first-order condition is

$$\digamma = E_t \sum_{k=0}^{\infty} \left(\alpha_w \alpha\right)^k \Psi_{t,t+k} \left[\frac{\sigma-1}{\sigma}(1+\tau)\,\bar{\pi}^k P_t(f) - \frac{\bar{\pi}^k W(f)}{MPL_{t+k}(f)}\right] Y_{t+k}(f) = 0. \tag{10}$$

Note that the only difference between (10) and equation (8) in Erceg, Henderson, and Levin (2000) is that the probability of an unchanged price is $\alpha_w \alpha$.

To derive labor demand elasticity, first note that we have

$$\frac{dP_t(f)}{dW(f)} = -\frac{\digamma_W}{\digamma_p} = (1-\gamma)\frac{P_t(f)}{W(f)} \tag{11}$$

[1] Note that, from (6) and (7) it follows that

$$MC_t(f) = \frac{W_t(f)}{MPL_t(f)}.$$

For future reference, note that from (5) it follows that

$$\frac{\partial L_t(f)}{\partial W(f)} = -\gamma \frac{L_t(f)}{W(f)} + L_t(f) \frac{\partial Y_t(f)}{\partial W(f)} \frac{1}{Y_t(f)}, \tag{12}$$

where $\frac{\partial Y_t(f)}{\partial W_t(f)} = \frac{\partial Y_t(f)}{\partial P_t(f)} \frac{\partial P_t(f)}{\partial W_t(f)}$ and thus, using (11) and (3) we have that

$$\frac{\partial L_t(f)}{\partial W(f)} = -(\gamma + \sigma(1-\gamma)) \frac{L_t(f)}{W(f)}. \tag{13}$$

The wage elasticity of labor demand is given by

$$\varepsilon_L = \frac{\partial L_t(f)}{\partial W(f)} \frac{W(f)}{L_t(f)} = -(\gamma + \sigma(1-\gamma)). \tag{14}$$

2.4 Households

The economy is populated by a continuum of households, indexed on the unit interval, which each supply labor to a single firm. The expected life-time utility of the household working at firm f in period t, is given by[2]

$$E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} \left[u(C_s(f), Q_s) + l\left(\frac{M_s(f)}{P_s}\right) - v(L_s(f), Z_s) \right] \right\}, \tag{15}$$

where period s utility is additively separable in three arguments, consumption $u(C_s(f), Q_s)$, where $C_s(f)$ is final goods consumption, subject to a consumption shock Q_s common to all households, real money balances $l\left(\frac{M_s(f)}{P_s}\right)$, where $M_s(f)$ denotes money holdings, and the disutility of working $v(L_s(f), Z_s)$, where $L_s(f)$ is the labor supply of the household f in period s, subject to a labor-supply shock Z_s common to all households. Finally, $\beta \in (0,1)$ is the households discount factor.

The budget constraint of the household is given by

$$\frac{\delta_{t+1,t} B_t(f)}{P_t} + \frac{M_t(f)}{P_t} + C_t(f) = \frac{M_{t-1}(f) + B_{t-1}(f)}{P_t} + (1+\tau_w) \frac{W_t(f) L_t(f)}{P_t} + \frac{\Gamma_t}{P_t} + \frac{T_t(f)}{P_t}. \tag{16}$$

The term $\delta_{t+1,t}$ represent the price vector of assets that pays one unit of currency in a particular state of nature in the subsequent period while the corresponding elements in $B_t(f)$ represent the quantity of such claims bought by the household. Thus, $B_{t-1}(f)$ denotes the realization of such claims bought in the previous period. Moreover, $W_t(f)$ denotes the households nominal wage and τ_w is the tax rate (subsidy) on labor income. Each household own an equal share of all firms and of the aggregate capital stock. Then, Γ_t is the household's aliquot share of profits and rental income. Finally, $T_t(f)$

[2] Note that Q_s and Z_s are held constant in the paper.

denotes nominal lump sum transfers from the government. We assume that there exists complete contingent claims markets (except for leisure) and as well as an equal initial wealth across households. Then households are homogeneous with respect to consumption and money holdings, i.e., we have $C_t(f) = C_t$, and $M_t(f) = M_t$ for all t.

The value function corresponding to the consumer maximization problem is

$$V(B_{t-1}, M_{t-1}) = \max E_t \left\{ u(C_t, Q_t) + l\left(\frac{M_t}{P_t}\right) - v(L_t(f), Z_t) + \beta V(B_t, M_t) \right\} \tag{17}$$

subject to

$$\frac{\delta_{t+1,t} B_t}{P_t} + \frac{M_t}{P_t} + C_t \leq \frac{M_{t-1} + B_{t-1}}{P_t} + (1 + \tau_w) \frac{W_t(f) L_t(f)}{P_t} + \frac{\Gamma_t}{P_t} + \frac{T_t(f)}{P_t}. \tag{18}$$

Using the envelope theorem to compute V_M and V_B and the first-order conditions with respect to C_t and B_t to derive the Euler equation, we arrive at the following expressions

$$l_{\frac{M}{P}}\left(\frac{M_s}{P_s}\right) = \lambda_t \frac{1}{P_t} - \beta E_t \left(u_C(C_{t+1}, Q_{t+1}) \frac{1}{P_{t+1}} \right), \tag{19}$$

$$u_C(C_t, Q_t) = \beta E_t \left(u_C(C_{t+1}, Q_{t+1}) R_t \right), \tag{20}$$

where

$$R_t = \frac{1}{E_t \delta_{t+1,t}} \frac{P_t}{P_{t+1}}, \tag{21}$$

is defined as the gross risk-free interest real rate and

$$I_t = \frac{1}{E_t \delta_{t+1,t}} \tag{22}$$

is the corresponding nominal interest rate.

For further use, let ρ_C and ρ_L denote the intertemporal elasticity of substitution in consumption and labor supply, respectively. That is

$$\rho_C = -\frac{\bar{u}_{CC} \bar{C}}{\bar{u}_C}, \tag{23}$$

$$\rho_L = -\frac{\bar{v}_{LL} \bar{L}}{\bar{v}_L}, \tag{24}$$

where $\bar{x}$ refers to the steady state value of the variable x.

2.4.1 Wages

Wages are determined in bargaining between firms and households. Since there is equivalence between the standard non-cooperative approach in Rubinstein (1982) and the Nash bargaining approach, we use the latter method. Let U_u^t and U_f^t denote the appropriate union and firm payoffs, respectively, and U_o the household outside option. The wage is then chosen such that is solves the following problem

$$\max_{W(f)} \left(U_u^t - U_o\right)^{\varphi} \left(U_f^t\right)^{1-\varphi}, \tag{25}$$

where φ denotes the bargaining power of households.

To state household utility U_u^t, we let

$$\Upsilon_{t,t+k} = u\left(C_{t+k}, Q_{t+k}\right) - v\left(L_{t,t+k}\left(f\right), Z_{t+k}\right), \tag{26}$$

denote per-period utility where $L_{t,t+k}\left(f\right)$ denotes labor demand in period $t+k$ when prices last were changed in period t. That labor demand depends on the period when prices last were reset is clear, since optimal prices vary over time due to different values of shocks, which in turn affect goods demand and hence labor demand. For a comprehensive analysis of this, see expression (97) below. Household utility U_u^t is

$$U_u^t = E_t \sum_{k=0}^{\infty} \left(\alpha_w \alpha \beta\right)^k \Upsilon_{t,t+k} + E_t \sum_{k=1}^{\infty} \left(\alpha_w \beta\right)^k (1-\alpha) \sum_{j=0}^{\infty} \left(\alpha_w \alpha \beta\right)^j \Upsilon_{t+k,t+k+j}. \tag{27}$$

Here, we use superscript to denote the period when the wage contract is changed, in order to distinguish it from the notation for price changes.

Let per-period real profit in period $t+k$ when prices last were changed in t be denoted as

$$\phi_{t,t+k}\left(W\left(f\right)\right) = (1+\tau) \frac{P_t\left(f\right)\bar{\pi}^k}{P_{t+k}} Y_{t+k}\left(f\right) - tc\left(w_{t+k}\left(f\right), p_{t+k}^c, Y_{t+k}\left(f\right)\right), \tag{28}$$

where $w_{t+k}\left(f\right) = \frac{\bar{\pi}^k W(f)}{P_{t+k}}$, the firm payoff U_f^t is

$$U_f^t = E_t \sum_{k=0}^{\infty} \left(\alpha_w \alpha\right)^k \psi_{t,t+k} \phi_{t,t+k}\left(W\left(f\right)\right) + E_t \sum_{k=1}^{\infty} \left(\alpha_w\right)^k (1-\alpha) \sum_{j=0}^{\infty} \left(\alpha_w \alpha\right)^j \psi_{t,t+k+j} \phi_{t+k,t+k+j}\left(W\left(f\right)\right). \tag{29}$$

To simplify notation, especially in section 4.4 below, we use gradient notation to indicate derivatives. For example, the partial derivative of the above expression with respect to the wage W is denoted

$$\nabla_W U_f^t. \tag{30}$$

The first-order condition corresponding to (25) is

$$\varphi U_f^t \nabla_W U_u^t + (1-\varphi)\left(U_u^t - U_o\right)\nabla_W U_f^t = 0. \tag{31}$$

Alternatively, we can write

$$\varphi \nabla_W U_u^t + (1-\varphi)\frac{U_u^t - U_o}{U_f^t}\nabla_W U_f^t = 0. \tag{32}$$

This is our counterpart to equation (16) in Erceg, Henderson, and Levin (2000). Here

$$\nabla_W U_u^t = E_t \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \nabla_W \Upsilon_{t,t+k} + E_t \sum_{k=1}^{\infty} (\alpha_w \beta)^k (1-\alpha) \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \nabla_W \Upsilon_{t+k,t+k+j}, \tag{33}$$

where, using (13), we have

$$\begin{aligned}\nabla_W \Upsilon_{t,t+k} &= u_C\left(C_{t+k}, Q_{t+k}\right)(1+\tau_w)\left(\frac{\bar{\pi}^k L_{t,t+k}(f)}{P_{t+k}} - \frac{\bar{\pi}^k W(f)}{P_{t+k}}(\gamma + \sigma(1-\gamma))\frac{L_{t,t+k}(f)}{W(f)}\right) \\ &\quad + (\gamma + \sigma(1-\gamma))\, v_L\left(L_{t,t+k}(f), Z_{t+k}\right)\frac{L_{t,t+k}(f)}{W(f)},\end{aligned} \tag{34}$$

Finally, $\nabla_W U_f^t$ can be written as

$$\nabla_W U_f^t = E_t \sum_{k=0}^{\infty} (\alpha_w \alpha)^k \psi_{t,t+k} \nabla_W \phi_{t,t+k} + E_t \sum_{k=1}^{\infty} (\alpha_w)^k (1-\alpha) \sum_{j=0}^{\infty} (\alpha_w \alpha)^j \psi_{t,t+k+j} \nabla_W \phi_{t+k,t+k+j}, \tag{35}$$

where

$$-\nabla_W \phi_{t,t+k} = (1-\gamma)\frac{tc\left(w_t(f), p_t^c, Y_t(f)\right)}{W(f)}, \tag{36}$$

and where we have used that the envelope theorem implies that all effects of a change in $W(f)$ on prices are eliminated.

2.5 Steady state

We now turn to the (non-stochastic) steady state of the model.[3] Note that the steady state of the real variables is the same in the flexible price model and the sticky price model. In the steady state, R, C, $Y(f)$ and B are constant. Moreover, $B = 0$. Also, M and P grows with the rate $\bar{\pi}$, i.e., we have $\frac{P_{t+1}}{P_t} = \bar{\pi}$ and $\bar{I} = \bar{R}\bar{\pi}$.

[3] That is, a situation where the disturbances Z_t, Q_t, A_t and G_t are equal to their mean values at all dates.

2.5.1 Prices

In steady state, the firm first-order condition (10) for price setting becomes

$$(1+\tau)\frac{\sigma-1}{\sigma}-\frac{\bar{w}}{\overline{MPL}}=0, \tag{37}$$

where $\bar{w}$ is the steady state real wage. Since we assume that monetary policy is used only to stabilize deviations from the flexible-price equilibrium, we require that τ is determined such that $(1+\tau)\frac{\sigma-1}{\sigma}=1$, i.e.

$$\tau=\frac{\sigma}{\sigma-1}-1=\frac{1}{\sigma-1}. \tag{38}$$

2.5.2 Wages

Real wages can be determined from the first-order condition for prices. From the choice of τ, we get that

$$\bar{w}=\overline{MPL}, \tag{39}$$

and, by the resource constraint and $G_t=\bar{G}=0$, we have

$$\bar{Y}(f)=\bar{Y}=\bar{C} \tag{40}$$

in the steady state.

Now, let us turn to the Nash bargaining solution in steady state. The first-order condition (31) then is

$$\varphi\bar{U}_f(W(f))\nabla_W\bar{U}_u(W(f))+(1-\varphi)\left(\bar{U}_u(W(f))-\bar{U}_o\right)\nabla_W\bar{U}_f(W(f)), \tag{41}$$

where $\bar{U}_u(W(f))$ etc. indicates that all variables except $W(f)$ are at steady state levels, noting that the steady state value of $\psi_{t,t+k}$ is $\bar{\psi}_k=\beta^k$. Using (13), (34), (36) and that the real total cost is, using $\overline{mc}=1$, $\overline{tc}=\bar{Y}$ gives that

$$\begin{aligned}\bar{\Upsilon} &= u\left(\bar{C},\bar{Q}\right)-v\left(\bar{L},\bar{Z}\right)=\bar{u}-\bar{v}, \\ \bar{\phi} &= (1+\tau)\bar{Y}-\overline{tc}=\tau\bar{Y},\end{aligned} \tag{42}$$

and letting

$$\bar{\Upsilon}_o=u\left(\bar{C}-b\bar{w}\bar{L},\bar{Q}\right)-v\left(0,\bar{Z}\right), \tag{43}$$

expression (41) can be written as

$$\varphi\tau\bar{Y}\left[\bar{u}_C\left(1+\tau_w\right)\left(1-\sigma\right)\bar{w}\left(1-\gamma\right)+\bar{v}_L\left(\gamma+\sigma\left(1-\gamma\right)\right)\right]\bar{L}-\left(1-\varphi\right)\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\left(1-\gamma\right)\bar{Y}=0. \quad (44)$$

2.5.3 Taxes and subsidies

We also need to adjust either the labor tax/subsidy τ_w, or the outside option U_o so that efficiency is achieved. When the labor tax is used we get using (44), using (38) that we from an efficient consumption choice have $\bar{u}_C\overline{MPL}=\bar{v}_L$ implying $\bar{v}_L\bar{L}=\bar{u}_C\bar{C}\left(1-\gamma\right)$ and that the labor cost share is $(1-\gamma)$ that

$$\tau_w=\frac{1}{\left(1-\sigma\right)\left(1-\gamma\right)}\left(\frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)-\left(\gamma+\sigma\left(1-\gamma\right)\right)\right)-1. \quad (45)$$

Note that τ_w increases in the bargaining power φ of workers. When taxes and the outside option are chosen to ensure efficiency, we can write

$$1+\tau_w=\frac{1}{1+\varepsilon_L}\frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)+\frac{\varepsilon_L}{1+\varepsilon_L}. \quad (46)$$

In general, taxes can be written

$$1+\tau_w=\varsigma_w+\frac{\varepsilon_L}{1+\varepsilon_L}. \quad (47)$$

The analysis below when the wage setting "Phillips" curve in section 4.4 can be easily modified to handle the case when inefficiencies are allowed. In particular, the wage setting "Phillips" curve depend on the same variables as in section 4.4 with different constants in expression (192).

2.5.4 Interest

From the Euler equation we get that

$$1=\frac{1}{\beta}\frac{1}{I}\bar{\pi} \quad (48)$$

or, in real terms, $\bar{R}=\frac{1}{\beta}$ and the nominal interest rate is then $\bar{I}=\frac{\bar{\pi}}{\beta}$.

2.5.5 Equilibrium

We have the following equations that determine the real variables in equilibrium of the economy in steady state. First, efficient consumption labor choice implies that

$$\bar{v}_L\left(\bar{L},\bar{Z}\right)\bar{L}=\bar{u}_C\left(\bar{C},\bar{Q}\right)\bar{C}\left(1-\gamma\right). \quad (49)$$

Note that this can be rewritten as $\bar{v}_L = \bar{u}_C \bar{w}$, using that from efficiency on labor market we have

$$\overline{MPL} = (1-\gamma)\frac{\bar{Y}}{\bar{L}} = \bar{w}. \tag{50}$$

The reason why (44) does not enter in the two expressions above is that τ_w and/or $\bar{U}_o$ is used to ensure that the wage bargain leads to an efficient outcome. Second, from goods market efficiency we have

$$\overline{mc} = \Gamma\frac{1}{\bar{A}}(\bar{w})^{1-\gamma}(\bar{p}^c)^{\gamma} = 1. \tag{51}$$

Third, optimal capital choice gives

$$\bar{K} = \Gamma\gamma\frac{\bar{Y}}{\bar{A}}(\bar{w})^{1-\gamma}(\bar{p}^c)^{\gamma-1}. \tag{52}$$

Since $\bar{Z}$, $\bar{Q}$, $\bar{A}$, $\bar{K}$, γ and Γ are parameters of the problem, we have six equations and six unknowns. The problem can be simplified by first, combining the second and third equation to get

$$\bar{p}^c = \gamma\frac{\bar{Y}}{\bar{K}}. \tag{53}$$

Then, using $\bar{C} = \bar{Y}$ in (49) and (50) with the expression above in (51) gives the following equation system

$$\begin{aligned}
\bar{v}_L\left(\bar{L},\bar{Z}\right)\bar{L} &= \bar{u}_C\left(\bar{Y},\bar{Q}\right)\bar{Y}(1-\gamma), \\
1 &= \Gamma\frac{1}{\bar{A}}\left((1-\gamma)\frac{\bar{Y}}{\bar{L}}\right)^{1-\gamma}\left(\gamma\frac{\bar{Y}}{\bar{K}}\right)^{\gamma},
\end{aligned} \tag{54}$$

consisting of the two unknowns $\bar{L}$ and $\bar{Y}$. We determine $\bar{L}$ by using the first expression together with the definition of the marginal product (50) and the technology (4)

$$\frac{\bar{v}_L\left(\bar{L},\bar{Z}\right)}{\bar{u}_C\left(\bar{A}\bar{K}^{\gamma}\bar{L}^{1-\gamma},\bar{Q}\right)} = (1-\gamma)\bar{A}\bar{K}^{\gamma}\bar{L}^{-\gamma}. \tag{55}$$

Again, taxes are determined from (38) and (45).

3 Loglinearizing the flexible price equilibrium

Now, let us log-linearize the model around the steady state. We first do this at the flexible price and wage equilibrium. This is then used to derive the log-linearization for the sticky price and wage equilibrium in terms of deviations from flexible-price variables. Here, we focus on a limiting cashless economy.

Let X^* denote the value of a variable in the flexible-price equilibrium.

3.1 Euler Equation

To find the Euler equation we use the definition (23) and log-linearize expression (20). We get

$$\hat{C}_t^* + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{CC}\bar{C}}\hat{Q}_t = E_t\left(\hat{C}_{t+1}^* + \frac{\bar{u}_{CQ}\bar{Q}}{\bar{u}_{CC}\bar{C}}\hat{Q}_{t+1} - \frac{1}{\rho_C}\hat{R}_t^*\right). \tag{56}$$

3.2 Prices, real wages and output

Rewriting problem (9) when $\alpha = 0$, we can find $P_t^*(f)$ by maximizing

$$\max_{P_t(f)} E_t\left((1+\tau)P_t^*(f)Y_t^*(f) - MC_t^* Y_t^*(f)\right) \tag{57}$$

$$st.\ Y_t^*(f) = \left(\frac{P_t^*(f)}{P_t^*}\right)^{-\sigma} Y_t^*.$$

Using that $\frac{\sigma}{(1+\tau)(\sigma-1)} = 1$, that $MC_t^* = \frac{W_t^*}{MPL_t^*}$, and that firms choose the same prices and face the same wages in flexible price equilibrium gives

$$P_t^*(f) = MC_t^* = \frac{W_t^*}{MPL_t^*} \iff \frac{W_t^*}{P_t^*} = MPL_t^*. \tag{58}$$

Log-linearizing gives, using the production function

$$\hat{w}_t^* = \widehat{mpl}_t^* = \hat{A}_t - \gamma\hat{L}_t^*. \tag{59}$$

Also, log-linearizing the production function $Y_t^* = A_t\bar{K}^\gamma L_t^*(f)^{1-\gamma}$ gives

$$\hat{L}_t^* = \frac{1}{1-\gamma}\left(\hat{Y}_t^* - \hat{A}_t\right). \tag{60}$$

Then, combining (59) and (60) gives

$$\hat{w}_t^* = \frac{1}{1-\gamma}\hat{A}_t - \frac{\gamma}{1-\gamma}\hat{Y}_t^*. \tag{61}$$

3.3 Output and Productivity

Due to the tax scheme and flexible prices and wages we have

$$u_C(C_t^*, Q_t)\,MPL_t^* = v_L(L_t^*, Z_t), \tag{62}$$

in equilibrium.

Using that $L_t^* = (1-\gamma)\frac{Y_t^*}{MPL_t^*}$, log-linearizing and using that $\hat{Y}_t^* = \hat{C}_t^* + G_t$ gives[4]

$$\begin{aligned}\left(\bar{u}_{CC}\bar{C}^* - \frac{(1-\gamma)\bar{Y}^*}{\overline{MPL}^2}\bar{v}_{LL}\right)\hat{Y}_t^* &= -\bar{u}_{CQ}\bar{Q}\hat{Q}_t + \bar{u}_{CC}\bar{C}^*G_t + \frac{\bar{Z}}{\overline{MPL}}\bar{v}_{LZ}\hat{Z}_t \\ &\quad -\frac{1}{\overline{MPL}}\left(\bar{v}_L + \bar{v}_{LL}\frac{(1-\gamma)\bar{Y}^*}{\overline{MPL}}\right)\widehat{mpl}_t^*. \end{aligned} \tag{63}$$

Thus, we have now $\hat{Y}_t^*$ expressed in terms of shocks and $\widehat{mpl}_t^*$.

3.4 Wages

Recall that the wage is chosen to solve (25). Using that

$$\begin{aligned} U_f^t &= \left((1+\tau)Y_t^* - tc\left(w_t^*, p_t^{c*}, Y_t^*(f)\right)\right), \\ U_u^t &= u\left(C_t^*, Q_t^*\right) - v\left(L_t^*, Z_t\right), \end{aligned} \tag{64}$$

together with (13) gives

$$\begin{aligned} \nabla_W U_u^t &= u_C\left(C_t^*, Q_t^*\right)\frac{1+\tau_w}{P_t^*}\left(L_t^* + W_t^*\frac{\partial L_t^*}{\partial W}\right) - v_L\left(L_t^*, Z_t\right)\frac{\partial L_t^*}{\partial W}, \\ \nabla_W U_f^t &= -\frac{1}{P_t^*}(1-\gamma)\frac{tc\left(w_t^*, p_t^{c*}, Y_t^*\right)}{W_t^*}. \end{aligned} \tag{65}$$

The first-order condition (31) can then be written as

$$\begin{aligned} &\varphi\left((1+\tau)Y_t^* - tc\left(w_t^*, p_t^{c*}, Y_t^*\right)\right) \\ &\times\left(u_C\left(C_t^*, Q_t^*\right)(1+\tau_w)w_t^*(1+\varepsilon_L)L_t^* - \varepsilon_L v_L\left(L_t^*, Z_t\right)L_t^*\right) \\ &-(1-\varphi)\left(u\left(C_t^*, Q_t^*\right) - v\left(L_t^*, Z_t\right) - \bar{\Upsilon}_o\right)(1-\gamma)\,tc\left(w_t^*, p_t^{c*}, Y_t^*\right). \end{aligned} \tag{66}$$

3.5 Interest

The relationship between nominal and real interest rates is derived from $R_t = \frac{P_t}{P_{t+1}}I_t$. We have, using that $\pi_{t+1} = \frac{P_{t+1}}{P_t}$

$$\hat{I}_t - E_t\hat{\pi}_{t+1} = E_t\hat{R}_t. \tag{67}$$

[4] Note that G_t is held constant in the main paper.

3.6 Shocks and real wages

To derive an expression that relates real wages to shocks, we first define the constant Λ^* as

$$\Lambda^* = \bar{u}_C \left(-\rho_C + \rho_L \frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma} \right). \tag{68}$$

Using (61), (63), (23), (24) and that $\hat{w}_t^* = \widehat{mpl}_t^*$ to write $\hat{w}_t^*$ in terms of shocks only gives

$$\hat{w}_t^* = \frac{1}{\Lambda^*} \left(\frac{\bar{u}_C \left(-\rho_C + \rho_L\right)}{1-\gamma} \hat{A}_t + \frac{\gamma}{1-\gamma} \left(\bar{u}_{CQ} \bar{Q} \hat{Q}_t - \bar{u}_{CC} \bar{C}^* G_t - \frac{\bar{Z}}{\overline{MPL}} \bar{v}_{LZ} \hat{Z}_t \right) \right). \tag{69}$$

Since $\Lambda^* < 0$ and $-\rho_C + \rho_L < 0$, the coefficient in front of $\hat{A}_t$ is positive and the coefficient in front of G_t is negative. The coefficients in front of $\hat{Q}_t$ and $\hat{Z}_t$ depend on the cross derivatives of u and v. If $\bar{u}_{CQ}$ is positive and $\bar{v}_{LZ}$ is negative as in Erceg, Henderson, and Levin (2000), the coefficient in front of $\hat{Q}_t$ and $\hat{Z}_t$ are negative. Note that, in terms of the notation in the main text, we have

$$\begin{aligned} a_Q &= \frac{1}{\Lambda^*} \frac{\gamma}{1-\gamma} \bar{u}_{CQ} \bar{Q} < 0, \\ a_Z &= -\frac{1}{\Lambda^*} \frac{\gamma}{1-\gamma} \frac{\bar{Z}}{\overline{MPL}} \bar{v}_{LZ} < 0, \\ a_A &= \frac{1}{\Lambda^*} \frac{\bar{u}_C \left(-\rho_C + \rho_L\right)}{1-\gamma} > 0, \\ a_G &= -\frac{1}{\Lambda^*} \frac{\gamma}{1-\gamma} \bar{u}_{CC} \bar{C}^* < 0. \end{aligned} \tag{70}$$

To simplify analysis we suppress the shocks $\hat{Q}_t$, $\hat{Z}_t$, $\hat{A}_t$ and G_t and assume that $\hat{w}_t^*$ follows an AR(1) process

$$\hat{w}_t^* = \eta \hat{w}_{t-1}^* + \varepsilon_t. \tag{71}$$

4 Loglinearizing the sticky price equilibrium

Now, let us loglinearize the model with sticky prices. As above, we start by loglinearizing the Euler equation.

4.1 Euler and the IS equation

Log-linearizing expression (20) gives, using that $\hat{C}_t = \hat{Y}_t + G_t$ and $\hat{C}_t^* = \hat{Y}_t^* + G_t$

$$\hat{Y}_t - \hat{Y}_t^* = E_t \left(\hat{Y}_{t+1} - \hat{Y}_{t+1}^* - \frac{1}{\rho_C} \left(\hat{I}_t - \hat{\pi}_{t+1} - \hat{R}_t^* \right) \right). \tag{72}$$

4.2 Loglinearization of some real and nominal variables

Before we can proceed to loglinearize the price and wage setting decisions, we need to loglinearize some other variables in the model.

4.2.1 Marginal product

To derive an expression for the marginal product, we first loglinearize the production function as

$$\hat{Y}_t(f) = \hat{A}_t + (1-\gamma)\hat{L}_t(f) + \gamma \hat{K}(f). \tag{73}$$

Loglinearizing expression (7) and aggregating over firms gives

$$\widehat{mpl}_t = \hat{A}_t + \gamma \hat{w}_t - \gamma \hat{p}_t^c, \tag{74}$$

where $\hat{w}_t = \int \hat{w}_t(f)\,df$ and $\widehat{mpl}_t = \int \widehat{mpl}_t(f)\,df$ is the aggregate real wage and marginal product, respectively.

Let us solve for the relative price of labor and capital. Note that capital is flexible and that all firms face the same price of capital. From above, all firms choose $K_t(f)$ as described by (5)

$$K_t(f)(p_t^c)^{1-\gamma} = \gamma \Gamma \frac{Y_t(f)}{A_t}(w_t(f))^{1-\gamma}. \tag{75}$$

Log-linearizing the above expression and integrating over all firms gives

$$\hat{w}_t - \hat{p}_t^c = \frac{1}{1-\gamma}\left(\hat{A}_t - \hat{Y}_t\right). \tag{76}$$

Using (76) in (74) gives

$$\widehat{mpl}_t = \frac{1}{1-\gamma}\left(\hat{A}_t - \gamma\hat{Y}_t\right). \tag{77}$$

To derive a relationship between $\widehat{mpl}_t$ and the (flexible price) real wage, we use expression (61) and hence, letting the output gap be denoted as

$$\hat{x}_t = \hat{Y}_t - \hat{Y}_t^*, \tag{78}$$

we get

$$\widehat{mpl}_t = \hat{w}_t^* - \frac{\gamma}{1-\gamma}\hat{x}_t. \tag{79}$$

4.2.2 Marginal rate of substitution

The marginal rate of substitution is defined as

$$MRS_t = \frac{v_L\left(L_t\left(f\right), Z_t\right)}{u_C\left(C_t, Q_t\right)}. \tag{80}$$

Loglinearizing and integrating over all firms/unions, using that we from expressions (49) and (50) have $\overline{MPL} = \overline{MRS}$ and that $\hat{C}_t = \hat{Y}_t - G_t$, gives

$$\widehat{mrs}_t = -\frac{1}{\bar{u}_C}\bar{u}_{CC}\bar{C}\hat{Y}_t + \frac{1}{\bar{u}_C}\bar{u}_{CC}\bar{C}G_t - \frac{1}{\bar{u}_C}\bar{u}_{CQ}\bar{Q}\hat{Q}_t + \frac{1}{\overline{MPL}}\frac{\bar{v}_{LL}}{\bar{u}_C}\bar{L}E_f\hat{L}_t\left(f\right) + \frac{1}{\overline{MPL}}\frac{\bar{v}_{LZ}}{\bar{u}_C}\bar{Z}\hat{Z}_t. \tag{81}$$

Using that $Y_t\left(f\right) = \frac{1}{1-\gamma}MPL_t\left(f\right)L_t\left(f\right)$ and loglinearizing gives

$$E_f\hat{L}_t\left(f\right) = \hat{Y}_t - E_f\widehat{mpl}_t\left(f\right), \tag{82}$$

where E_f denotes the expectation taken over all firms. Letting

$$\Lambda = u_{CC}\left(\bar{C}^*, \bar{Q}\right)\bar{C}^* - \frac{\bar{L}}{\overline{MPL}}v_{LL}\left(\bar{L}, \bar{Z}\right), \tag{83}$$

$\widehat{mrs}_t$ can be rewritten as

$$\begin{aligned}\widehat{mrs}_t \quad = \quad & -\frac{1}{\bar{u}_C}\Lambda\hat{Y}_t + \frac{1}{\bar{u}_C}\bar{u}_{CC}\bar{C}G_t \\ & -\frac{1}{\bar{u}_C}\bar{u}_{CQ}\bar{Q}\hat{Q}_t - \frac{1}{\overline{MPL}}\frac{\bar{v}_{LL}}{\bar{u}_C}\bar{L}E_f\widehat{mpl}_t\left(f\right) + \frac{1}{\overline{MPL}}\frac{\bar{v}_{LZ}}{\bar{u}_C}\bar{Z}\hat{Z}_t.\end{aligned} \tag{84}$$

Subtracting flexible-price marginal rate of substitution and using that $\bar{u}_C\overline{MPL} = \bar{v}_L$ from (49) gives

$$\widehat{mrs}_t - \widehat{mrs}_t^* = -\frac{1}{\bar{u}_C}\Lambda\left(\hat{Y}_t - \hat{Y}_t^*\right) - \frac{\bar{v}_{LL}}{\bar{v}_L}\bar{L}\left(E_f\widehat{mpl}_t\left(f\right) - \widehat{mpl}_t^*\right) \tag{85}$$

Using expression (79), $\widehat{mrs}_t^* = \widehat{mpl}_t^* = \hat{w}_t^*$ (from our use of τ, τ_w and U_o) and that, using (83), we have

$$\frac{1}{\bar{u}_C}\Lambda - \frac{v_{LL}}{v_L}\bar{L}\frac{\gamma}{1-\gamma} = \frac{\bar{u}_{CC}\bar{C}^*}{\bar{u}_C} - \frac{\bar{v}_{LL}}{\bar{v}_L}\bar{L}\frac{1}{1-\gamma} = -\rho_C + \frac{1}{1-\gamma}\rho_L, \tag{86}$$

expression (85) can be rewritten as

$$\widehat{mrs}_t = \hat{w}_t^* + \left(\rho_C - \rho_L\frac{1}{1-\gamma}\right)\hat{x}_t. \tag{87}$$

4.2.3 Relative prices and goods demand

We define the firms relative prices and wages as

$$
\begin{aligned}
q_t(f) &= \frac{P_t(f)}{P_t}, \\
n_t(f) &= \frac{W(f)}{W_t}, \\
w_t(f) &= \frac{W(f)}{P_t},
\end{aligned} \tag{88}
$$

and also

$$
\begin{aligned}
X_{t,k} &= \frac{\bar{\pi}^k P_t}{P_{t+k}} = \frac{\bar{\pi}^k}{\pi_{t+1} \cdot \ldots \cdot \pi_{t+k}}, \\
X_{t,k}^{\omega} &= \frac{\bar{\pi}^k W_t}{W_{t+k}} = \frac{\bar{\pi}^k}{\pi_{t+1}^{\omega} \cdot \ldots \cdot \pi_{t+k}^{\omega}}.
\end{aligned} \tag{89}
$$

The real wage at firm f at time $t+k$ is

$$
\frac{W(f)\,\bar{\pi}^k}{P_{t+k}} = \frac{W(f)\,\bar{\pi}^k}{W_{t+k}} w_{t+k} = n_t(f)\, X_{t,k}^{\omega} w_{t+k}. \tag{90}
$$

4.2.4 Output, labor demand, costs and profits

We first want to find a relationship between output and the prices of capital and labor, respectively. To eliminate the stock of capital from (73), we use that the ratio of the two expressions in (5) can be written as

$$
\hat{K}_t(f) = \hat{L}_t(f) - \hat{p}_t^c + \hat{w}_t(f). \tag{91}
$$

Then we can rewrite (73) as

$$
\hat{Y}_t(f) = \hat{A}_t + \hat{L}_t(f) + \gamma\left(\hat{w}_t(f) - \hat{p}_t^c\right). \tag{92}
$$

Consider the derivative of labor demand (13). Loglinearizing gives

$$
\frac{\widehat{\partial L_{t,t+k}(f)}}{\partial W(f)} = \hat{L}_{t,t+k}(f). \tag{93}
$$

Loglinearizing goods demand using (3) and (88) gives

$$
\hat{Y}_{t+k}(f) = -\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k} \hat{\pi}_{t+l}\right) + \hat{Y}_{t+k}. \tag{94}
$$

Loglinearizing total costs (8), using a loglinearization of (88) and (90), we get

$$\begin{aligned}\widehat{tc}\left(\frac{W(f)\,\bar{\pi}^k}{P_{t+k}}, p^c_{t+k}, Y_{t+k}(f)\right) &= (1-\gamma)\left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}^{\omega}_{t+l} + \hat{w}_{t+k}\right) - \hat{A}_{t+k} + \gamma\hat{p}^c_{t+k} \qquad (95)\\ &\quad -\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \hat{Y}_{t+k}.\end{aligned}$$

To loglinearize labor demand, we use goods demand (5). Inserting real prices in (5) gives

$$\hat{L}_{t,t+k}(f) = -\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \hat{Y}_{t+k} - \hat{A}_{t+k} - \gamma\left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}^{\omega}_{t+l} + \hat{w}_{t+k}\right) + \gamma\hat{p}^c_{t+k}. \qquad (96)$$

Rewriting $\hat{L}_{t,t+k}(f)$ in terms of relative prices and wages only, using (76), gives

$$\hat{L}_{t,t+k}(f) = -\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\left(\hat{Y}_{t+k} - \hat{A}_{t+k}\right) - \gamma\left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}^{\omega}_{t+l}\right). \qquad (97)$$

Total costs can be rewritten, using (96) and goods demand, as

$$\widehat{tc}\left(\frac{W(f)\,\bar{\pi}^k}{P_{t+k}}, p^c_{t+k}, Y_{t+k}(f)\right) = \hat{L}_{t,t+k}(f) + \left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}^{\omega}_{t+l}\right) + \hat{w}_{t+k}. \qquad (98)$$

The loglinearized version of per period profits (28) is, using (94), (98), (97) and, that marginal cost being equal to one together with $\overline{tc} = \bar{Y}$

$$\begin{aligned}\bar{\phi}\hat{\phi}_{t,t+k} &= (1+\tau)\,\bar{Y}\,(1-\sigma)\left(\hat{q}_t - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + (1+\tau)\,\bar{Y}\hat{Y}_{t+k} \qquad (99)\\ &\quad -\bar{Y}\left(-\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\left(\hat{Y}_{t+k} - \hat{A}_{t+k}\right) + (1-\gamma)\left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}^{\omega}_{t+l}\right) + \hat{w}_{t+k}\right).\end{aligned}$$

Also, we have

$$\nabla_W\phi_{t,t+k} = -\frac{1}{P_{t+k}}\frac{\partial TC\left(\bar{\pi}^k W(f), Y_{t+k}(f)\right)}{\partial W(f)} = -(1-\gamma)\,\frac{tc\left(\frac{W(f)\bar{\pi}^k}{P_{t+k}}, p^c_{t+k}, Y_{t+k}(f)\right)}{W(f)}. \qquad (100)$$

We then get the loglinearized version of the derivative of per-period profits as

$$\begin{aligned}-\widehat{\nabla_W\phi}_{t,t+k} &= -\sigma\left(\hat{q}_t(f) - \sum_{l=1}^{k}\hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\left(\hat{Y}_{t+k} - \hat{A}_{t+k}\right) \qquad (101)\\ &\quad + (1-\gamma)\left(\hat{n}_t(f) - \sum_{l=1}^{k}\hat{\pi}^{\omega}_{t+l}\right) + \hat{w}_{t+k}.\end{aligned}$$

4.2.5 Utility

Loglinearizing per period union utility (26) gives

$$\bar{\Upsilon}\hat{\Upsilon}_{t,t+k} = \bar{u}\hat{u}_{t+k} - \bar{v}\hat{v}_{t+k}, \tag{102}$$

where

$$\begin{aligned} \bar{u}\hat{u}_{t+k} &= \bar{u}_C\bar{C}\hat{C}_{t+k} + \bar{u}_Q\bar{Q}\hat{Q}_{t+k}, \\ \bar{v}\hat{v}_{t+k} &= \bar{v}_L\bar{L}\hat{L}_{t,t+k}(f) + \bar{v}_Z\bar{Z}\hat{Z}_{t+k}. \end{aligned} \tag{103}$$

4.2.6 Total demand

The log-linear approximation of total demand is

$$\hat{Y}_t = \hat{C}_t + \left(G_t - \bar{G}\right). \tag{104}$$

4.3 Optimal Prices and the New Keynesian Phillips curve

The first-order condition for optimal price choices, i.e. expression (10), can be rewritten as

$$E_t \sum_{k=0}^{\infty} (\alpha_w \alpha)^k \Psi_{t,t+k} \left(q_t(f) X_{t,k} - mc_{t+k}(f)\right) Y_{t+k}(f) = 0, \tag{105}$$

where $mc_{t+k}(f)$ is the real marginal cost. Loglinearizing around steady state, and using that $\bar{\Psi}_k = P_t(\bar{\pi}\beta)^k$ (the value of the steady state path of $\Psi_{t,t+k}$, given an initial price level P_t), that $P_t \neq 0$ and that the probability that wages not open for renegotiation in period $t+k$ is $\alpha_w\alpha$ gives[5]

$$0 = E_t \sum_{k=0}^{\infty} (\alpha_w \alpha\beta)^k \left(\hat{q}_t(f) + \hat{X}_{t,k} - \widehat{mc}_{t+k}(f)\right) \bar{Y}. \tag{106}$$

Now, let us derive the aggregate supply equation (i.e., new Keynesian Phillips curve). Loglinearizing $X_{t,k}$ in (89) gives

$$-\sum_{k=0}^{\infty} (\alpha_w \alpha\beta)^k E_t\hat{X}_{t,k} = \sum_{l=1}^{\infty} \frac{(\alpha_w\alpha\beta)^l}{1-\alpha_w\alpha\beta} E_t\hat{\pi}_{t+l}. \tag{107}$$

Note that the wage distribution of the firms that change prices is not the same as for the entire population of firms. Let W_t^o denote the solution to problem (25). The average wage for those firms

[5] Note that $\Psi_{t,t+k} = \psi_{t,t+k}P_{t+k}$. Also, we have $\psi_{t,t+k} = \prod_{s=1}^{k} \psi_{t+s-1,t+s}$. Also, we normalize $\psi_{t,t} = 1$ as in Woodford (2003) page 68.

that change prices is then

$$W_t^p = \frac{(1-\alpha)\,\alpha_w}{(1-\alpha)\,\alpha_w + (1-\alpha_w)} \int \bar{\pi} W_{t-1}(f)\, df + \frac{(1-\alpha_w)}{(1-\alpha)\,\alpha_w + (1-\alpha_w)} \int W_t^o df. \tag{108}$$

The entire wage distribution evolves according to

$$W_t = \alpha_w \int \bar{\pi} W_{t-1}(f)\, df + (1-\alpha_w) \int W_t^o df. \tag{109}$$

Using (109) in (108), we get, in real terms

$$w_t^p = \frac{w_t}{(1-\alpha)\,\alpha_w + (1-\alpha_w)} \left(1 - \alpha_w \alpha \frac{\bar{\pi}}{\pi_t^\omega} \right). \tag{110}$$

Loglinearizing (110) gives, evaluating the real wage in $t+k$, and hence taking into account the effects of inflation on the real wage through $\hat{X}_{t,k}$

$$\hat{w}_{t+k}^p = \hat{w}_t + \frac{\alpha_w \alpha}{1-\alpha_w \alpha} \hat{\pi}_t^\omega + \hat{X}_{t,k}. \tag{111}$$

Deriving real marginal cost from the total cost expression in (8) and loglinearizing gives

$$\widehat{mc}_{t+k}(f) = (1-\gamma)\,\hat{w}_{t+k}^p(f) + \gamma \hat{p}_{t+k}^c - \hat{A}_{t+k}, \tag{112}$$

where $\hat{w}_{t+k}^p(f)$ is the loglinearized real wage for firms that change prices in t. Note that the average marginal cost for firms that change prices is, using the above expression (112) and expression (111)

$$\widehat{mc}_{t+k} = -\hat{A}_{t+k} + (1-\gamma) \left(\hat{w}_t + \frac{\alpha_w \alpha}{1-\alpha_w \alpha} \hat{\pi}_t^\omega + \hat{X}_{t,k} \right) + \gamma \hat{p}_{t+k}^c. \tag{113}$$

Expression (10) can be rewritten, aggregating over all firms that change prices and using (107)

$$\begin{aligned} 0 \;=\; & \frac{1}{1-\alpha_w\alpha\beta} \left(\hat{q}_t - (1-\gamma) \left(\hat{w}_t + \frac{\alpha_w \alpha}{1-\alpha_w \alpha} \hat{\pi}_t^\omega \right) \right) - E_t \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \left(\gamma \hat{p}_{t+k}^c - \hat{A}_{t+k} \right) \\ & -\gamma \sum_{k=1}^{\infty} \frac{(\alpha_w \alpha \beta)^k}{1-\alpha_w \alpha \beta} E_t \hat{\pi}_{t+k}. \end{aligned} \tag{114}$$

To write the expression above in terms of inflation, we need to express the relative prices $\hat{q}_t(f)$ in terms of inflation. To do this, we use the price evolution equation. Using that prices evolve according to

$$P_t = \alpha_w \alpha \int_0^1 \bar{\pi} P_{t-1}(f)\, df + (1-\alpha_w \alpha) \int_0^1 P_t^o(f)\, df, \tag{115}$$

gives, using $\frac{P_{t-1}}{P_t} = \frac{1}{\pi_t}$

$$1 = \alpha_w \alpha \left(\frac{\bar{\pi}}{\pi_t} \right)^{1-\sigma} + (1 - \alpha_w \alpha) \int (q_t(f))^{1-\sigma} df. \tag{116}$$

We thus have that

$$\hat{q}_t = \int \hat{q}_t(f)\, df = \frac{\alpha_w \alpha}{1 - \alpha_w \alpha} \hat{\pi}_t. \tag{117}$$

The first-order condition for price setting (106) can then be rewritten by using (117) in (114) in periods t and $t+1$, respectively, together with the real wage identity $\hat{w}_{t+1} = \hat{w}_t + \hat{\pi}^{\omega}_{t+1} - \hat{\pi}_{t+1}$

$$\begin{aligned} 0 \quad = \quad & \frac{\alpha_w \alpha}{1 - \alpha_w \alpha} \hat{\pi}_t - (1 - \gamma) \left(\hat{w}_t + \frac{\alpha_w \alpha}{1 - \alpha_w \alpha} \hat{\pi}^{\omega}_t \right) - (1 - \alpha_w \alpha \beta) \left(\gamma \hat{p}^c_t - \hat{A}_t \right) \\ & -\alpha_w \alpha \beta \left(\frac{1}{1 - \alpha_w \alpha} E_t \hat{\pi}_{t+1} - (1 - \gamma) \left(\hat{w}_t + \frac{1}{1 - \alpha_w \alpha} \hat{\pi}^{\omega}_{t+1} \right) \right). \end{aligned} \tag{118}$$

To eliminate capital prices from the above expression, we use (76) and (61) to get

$$\gamma \hat{p}^c_t - A_t = \gamma \hat{w}_t + \frac{\gamma}{1 - \gamma} \hat{x}_t - \hat{w}^*_t. \tag{119}$$

Then, defining

$$\Pi = \frac{1 - \alpha_w \alpha}{\alpha_w \alpha} (1 - \alpha_w \alpha \beta), \tag{120}$$

the first-order condition for price setting, or equivalently, the New Keynesian Phillips curve, is

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + (1 - \gamma) \left(\hat{\pi}^{\omega}_t - \beta E_t \hat{\pi}^{\omega}_{t+1} \right) + \Pi (\hat{w}_t - \hat{w}^*_t) + \frac{\gamma}{1 - \gamma} \Pi \hat{x}_t. \tag{121}$$

The only difference with expression (T1.4) in Erceg, Henderson, and Levin (2000) is the presence of the term involving wage inflation. Using (79) we can rewrite (121) as

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + (1 - \gamma) \left(\hat{\pi}^{\omega}_t - \beta E_t \hat{\pi}^{\omega}_{t+1} \right) + \Pi \left(\hat{w}_t - \widehat{mpl}_t \right). \tag{122}$$

4.3.1 Relationship between relative prices and wages

To analyze wage setting, we need to relate the relative prices to relative wages for the price adjusting firms (see the section 4.4 below on wage determination). Let us first look at the relationship between relative prices and wages for firms that changed wages in t and prices in $t+k$. The first order condition for price setting (10) is, where $\hat{q}^t_{t+k}$ is the loglinearized relative price in $t+k$ for firms that renegotiated their wages in t, and $\hat{n}^t$ the relative wage for firms that renegotiated their wages last in period t

$$0 = E_t \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \left(\hat{q}^t_{t+k} + \hat{X}_{t+k,k+j} - \widehat{mc}_{t+k+j}(f) \right) \bar{Y}, \tag{123}$$

where, deriving marginal cost from the expression (8) for total costs, and using that

$$w_{t+k}(f) = \frac{\bar{\pi}^k W_t(f)}{P_{t+k}} = \frac{W_t(f)}{W_t}\frac{W_t}{W_{t+k}}\frac{W_{t+k}}{P_{t+k}}, \tag{124}$$

we have

$$\widehat{mc}_{t+k+j}(f) = \hat{w}_{t+k+j} + (1-\gamma)\left(\hat{n}^t + \hat{X}^{\omega}_{t,k+j}\right) - \left(\frac{1}{1-\gamma}\hat{A}_{t+k+j} - \frac{\gamma}{1-\gamma}\hat{Y}_{t+k+j}\right). \tag{125}$$

Using (61) gives, where $\hat{x}$ denotes the output gap

$$\widehat{mc}_{t+k+j}(f) = \hat{w}_{t+k+j} - \hat{w}^*_{t+k+j} + (1-\gamma)\left(\hat{n}^t + \hat{X}^{\omega}_{t,k+j}\right) + \frac{\gamma}{1-\gamma}\hat{x}_{t+k+j}. \tag{126}$$

Rewriting the sums over $\hat{X}_{t+k,k+j}$ and $\hat{X}^{\omega}_{t,k+j}$ in expression (123) gives

$$\begin{aligned} -\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{X}_{t+k,k+j} &= \sum_{l=1}^{\infty}\frac{(\alpha_w\alpha\beta)^l}{1-\alpha_w\alpha\beta}E_t\hat{\pi}_{t+k+l}, \\ -\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{X}^{\omega}_{t,k+j} &= \frac{1}{1-\alpha_w\alpha\beta}\left(\sum_{l=1}^{k}E_t\hat{\pi}^{\omega}_{t+l} + \sum_{l=1}^{\infty}(\alpha_w\alpha\beta)^l E_t\hat{\pi}^{\omega}_{t+k+l}\right). \end{aligned} \tag{127}$$

Then, using the expression for $\widehat{mc}_{t+k+j}(f)$ in the first-order condition (123) we have

$$\begin{aligned} 0 &= \left(\hat{q}^t_{t+k} - (1-\gamma)\hat{n}^t\right) - (1-\alpha_w\alpha\beta)E_t\sum_{j=0}^{\infty}(\alpha_w\alpha\beta)^j\left(\hat{w}_{t+k+j} - \hat{w}^*_{t+k+j} + \frac{\gamma}{1-\gamma}\hat{x}_{t+k+j}\right) \\ &\quad -\sum_{j=1}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{\pi}_{t+k+j} + (1-\gamma)\left(\sum_{l=1}^{k}E_t\hat{\pi}^{\omega}_{t+l} + \sum_{j=1}^{\infty}(\alpha_w\alpha\beta)^j E_t\hat{\pi}^{\omega}_{t+k+j}\right). \end{aligned} \tag{128}$$

Leading one wage contract period ahead and combining gives

$$\hat{q}^t_{t+k} - E_t\hat{q}^{t+1}_{t+k} = (1-\gamma)\left(\hat{n}^t - E_t\hat{n}^{t+1} - E_t\hat{\pi}^{\omega}_{t+1}\right). \tag{129}$$

For the analysis of wages below, we also want to derive a relationship between relative prices in t and in $t+1$ for firms that last changed wages in period t. From using (128) when wages are renegotiated at t and prices at t and $t+1$, respectively, we have

$$\hat{q}^t_t - \alpha_w\alpha\beta\left(E_t\hat{q}^t_{t+1} + E_t\hat{\pi}_{t+1}\right) = (1-\alpha_w\alpha\beta)(1-\gamma)\hat{n}^t + (1-\alpha_w\alpha\beta)\left(\hat{w}_t - \hat{w}^*_t + \frac{\gamma}{1-\gamma}\hat{x}_t\right). \tag{130}$$

4.4 Optimal Wages and the wage setting "Phillips" Curve

Here, it is important to distinguish the period when the wage contract last was rewritten, the period when the price last was changed and the current period. Therefore, we use the following notation $x^t_{t+k,t+k+j}$ to denote the value of variable x in period $t+k+j$ when the wage contract last was renegotiated in t, the price last was changed in $t+k$.

In this section we derive the wage setting "Phillips" curve from expression (31). Loglinearizing the first-order condition (31) gives

$$\begin{aligned} 0 &= \varphi \nabla_W \bar{U}_u \widehat{\nabla_W U^t_u} + (1-\varphi)\left(\frac{1}{\bar{U}_f}\nabla_W \bar{U}_f \bar{U}_u \hat{U}^t_u - \frac{\bar{U}_u - \bar{U}_o}{\left(\bar{U}_f\right)^2}\nabla_W \bar{U}_f \bar{U}_f \hat{U}^t_f\right) \\ &\quad + (1-\varphi)\frac{\bar{U}_u - \bar{U}_o}{\bar{U}_f}\nabla_W \bar{U}_f \widehat{\nabla_W U^t_f}. \end{aligned} \tag{131}$$

The four terms in the above expressions are [6]

$$\begin{aligned} \bar{U}_u \hat{U}^t_u &= E_t \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \, \hat{\Upsilon}^t_{t,t+k} \\ &\quad + E_t \sum_{k=1}^{\infty} (\alpha_w)^{k-1} (\alpha_w (1-\alpha)) \beta^k \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \, \hat{\Upsilon}^t_{t+k,t+k+j}, \end{aligned} \tag{132}$$

$$\begin{aligned} \bar{U}_f \hat{U}^t_f &= E_t \sum_{k=0}^{\infty} (\alpha_w \alpha)^k \, \bar{\psi}_k \bar{\phi} \hat{\phi}^t_{t,t+k} (W(f)) \\ &\quad + E_t \sum_{k=1}^{\infty} (\alpha_w)^{k-1} (\alpha_w (1-\alpha)) \sum_{j=0}^{\infty} (\alpha_w \alpha)^j \, \bar{\psi}_{k+j} \bar{\phi} \hat{\phi}^t_{t+k,t+k+j} (W(f)), \end{aligned} \tag{133}$$

$$\begin{aligned} \nabla_W \bar{U}_u \widehat{\nabla_W U}^t_u &= E_t \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}^t_{t,t+k} \\ &\quad + E_t \sum_{k=1}^{\infty} (\alpha_w)^{k-1} (\alpha_w (1-\alpha)) \beta^k \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \, \overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}^t_{t+k,t+k+j}, \end{aligned} \tag{134}$$

and

$$\begin{aligned} \nabla_W \bar{U}_f \left(-\widehat{\nabla_W U^t_f}\right) &= E_t \sum_{k=0}^{\infty} (\alpha_w \alpha)^k \, \bar{\psi}_k \overline{\nabla_W \phi} \left(-\widehat{\nabla_W \phi}^t_{t,t+k}\right) + \\ &\quad + E_t \sum_{k=1}^{\infty} (\alpha_w)^{k-1} (\alpha_w (1-\alpha)) \sum_{j=0}^{\infty} (\alpha_w \alpha)^j \, \bar{\psi}_{k+j} \overline{\nabla_W \phi} \left(-\widehat{\nabla_W \phi}^t_{t+k,t+k+j}\right). \end{aligned} \tag{135}$$

[6] The last summation in the expressions contain all future price changes during the present wage contract.

Leading the first-order condition for wages one period, multiplying with $\alpha_w\beta$ and taking the expectation at t gives

$$
\begin{aligned}
0 &= \varphi\nabla_W\bar{U}_u\left(\widehat{\nabla_W U_u^t}-\alpha_w\beta E_t\widehat{\nabla_W U_u^{t+1}}\right)+(1-\varphi)\frac{1}{\bar{U}_f}\nabla_W\bar{U}_f\bar{U}_u\left(\hat{U}_u^t-\alpha_w\beta E_t\hat{U}_u^{t+1}\right) \\
&\quad +(1-\varphi)\left(-\frac{\bar{U}_u-\bar{U}_o}{\left(\bar{U}_f\right)^2}\nabla_W\bar{U}_f\bar{U}_f\left(\hat{U}_f^t-\alpha_w\beta E_t\hat{U}_f^{t+1}\right)\right) \\
&\quad +(1-\varphi)\frac{\bar{U}_u-\bar{U}_o}{\bar{U}_f}\nabla_W\bar{U}_f\left(-\widehat{\nabla_W U_f^t}-\alpha_w\beta E_t\left(-\widehat{\nabla_W U_f^{t+1}}\right)\right)
\end{aligned}
\tag{136}
$$

Also, we need to distinguish the period where the wage contract last was rewritten for the terms $\hat{\phi}_{t+k,t+k+j}$ and $\hat{\Upsilon}_{t+k,t+k+j}$ as well as for the corresponding derivatives. As for firm payoff and union utility, we indicate the wage contract period with superscripts, i.e., we use the notation $\hat{\phi}^t_{t+k,t+k+j}$ and $\hat{\Upsilon}^t_{t+k,t+k+j}$.

First, consider the first term in (136). Using expression (133) we can write

$$
\begin{aligned}
&\bar{U}_f\left(\hat{U}_f^t-\alpha_w\beta E_t\hat{U}_f^{t+1}\right) \\
=\;&\bar{\phi}\hat{\phi}^t_{t,t}+E_t\sum_{k=1}^{\infty}\left(\alpha_w\alpha\beta\right)^k\bar{\phi}\left(\hat{\phi}^t_{t,t+k}-\hat{\phi}^t_{t+1,t+k}\right) \\
&+\alpha_w\beta\sum_{k=0}^{\infty}\left(\alpha_w\alpha\beta\right)^k\bar{\phi}\left(\hat{\phi}^t_{t+1,t+1+k}-\hat{\phi}^{t+1}_{t+1,t+1+k}\right) \\
&+E_t\sum_{k=2}^{\infty}\left(\alpha_w\beta\right)^{k-1}\left(\alpha_w-\alpha_w\alpha\right)\beta\sum_{j=0}^{\infty}\left(\alpha_w\alpha\beta\right)^j\bar{\phi}\left(\hat{\phi}^t_{t+k,t+k+j}-\hat{\phi}^{t+1}_{t+k,t+k+j}\right),
\end{aligned}
\tag{137}
$$

where, from (28), (3), (88) and

$$
\frac{\bar{\pi}^{k+j}W_t(f)}{P_{t+k+j}}=\frac{W_t(f)}{W_t}\frac{\bar{\pi}^{k+j}W_t}{W_{t+k+j}}\frac{W_{t+k+j}}{P_{t+k+j}}=n^t(f)\,w_{t+k+j}X^{\omega}_{t,k+j},
\tag{138}
$$

we have

$$
\begin{aligned}
\phi^t_{t+k,t+k+j}(W(f)) &= (1+\tau)\,q^t_{t+k}(f)\frac{\bar{\pi}^jP_{t+k}}{P_{t+k+j}}\left(q^t_{t+k}(f)\frac{\bar{\pi}^jP_{t+k}}{P_{t+k+j}}\right)^{-\sigma}Y_{t+k+j} \\
&\quad -\Gamma\frac{1}{A_{t+k+j}}\left(n^t(f)\,w_{t+k+j}X^{\omega}_{t,k+j}\right)^{1-\gamma}\left(p^c_{t+k+j}\right)^{\gamma}\left(q^t_{t+k}(f)\frac{\bar{\pi}^jP_{t+k}}{P_{t+k+j}}\right)^{-\sigma}Y_{t+k+j}.
\end{aligned}
\tag{139}
$$

Loglinearizing gives, using that wages for firms that change wage contracts within the same time period are the same

$$
\begin{aligned}
\bar{\phi}\hat{\phi}^{t}_{t+k,t+k+j} &= \bar{Y}\left(\left(1+\tau\right)\left(1-\sigma\right)+\sigma\right)\hat{q}^{t}_{t+k}-\bar{Y}\left(1-\gamma\right)\hat{n}^{t}+R^{f,t}_{t+k,t+k+j}, \\
\bar{\phi}\hat{\phi}^{t+1}_{t+k,t+k+j} &= \bar{Y}\left(\left(1+\tau\right)\left(1-\sigma\right)+\sigma\right)\hat{q}^{t+1}_{t+k}-\bar{Y}\left(1-\gamma\right)\hat{n}^{t+1}+R^{f,t+1}_{t+k,t+k+j},
\end{aligned}
\tag{140}
$$

where

$$
\begin{aligned}
R^{f,t}_{t+k,t+k+j} &= \left(1+\tau\right)\bar{Y}\left(-\left(1-\sigma\right)\sum_{l=k+1}^{k+j}\hat{\pi}_{t+l}+\hat{Y}_{t+k+j}\right) \\
&\quad -\bar{Y}\left(\sigma\sum_{l=k+1}^{k+j}\hat{\pi}_{t+l}+\left(1-\gamma\right)\hat{w}_{t+k+j}+\left(\hat{Y}_{t+k+j}-\hat{A}_{t+k+j}\right)+\gamma\hat{p}^{c}_{t+k+j}-\left(1-\gamma\right)\sum_{l=1}^{k+j}\hat{\pi}^{\omega}_{t+l}\right),
\end{aligned}
\tag{141}
$$

and

$$
\begin{aligned}
R^{f,t+1}_{t+k,t+k+j} &= \left(1+\tau\right)\bar{Y}\left(-\left(1-\sigma\right)\sum_{l=k+1}^{k+j}\hat{\pi}_{t+l}+\hat{Y}_{t+k+j}\right) \\
&\quad -\bar{Y}\left(\sigma\sum_{l=k+1}^{k+j}\hat{\pi}_{t+l}+\left(1-\gamma\right)\hat{w}_{t+k+j}+\left(\hat{Y}_{t+k+j}-\hat{A}_{t+k+j}\right)+\gamma\hat{p}^{c}_{t+k+j}-\left(1-\gamma\right)\sum_{l=2}^{k+j}\hat{\pi}^{\omega}_{t+l}\right).
\end{aligned}
\tag{142}
$$

From (129) above, using that $(1+\tau)=\frac{\sigma}{\sigma-1}$ and the definition of $R^{f,t}_{t+k,t+k+j}$ and $R^{f,t+1}_{t+k,t+k+j}$ gives

$$
\begin{aligned}
\bar{\phi}\hat{\phi}^{t}_{t,t} &= -\bar{Y}\left(1-\gamma\right)\hat{n}^{t}+R^{f,t}_{t,t}, \\
\bar{\phi}\hat{\phi}^{t}_{t,t+k}-\bar{\phi}\hat{\phi}^{t}_{t+1,t+k} &= 0, \\
\bar{\phi}\hat{\phi}^{t}_{t+k,t+k+j}-\bar{\phi}\hat{\phi}^{t+1}_{t+k,t+k+j} &= -\bar{Y}\left(1-\gamma\right)\left(\hat{n}^{t}-\hat{n}^{t+1}\right)+\bar{Y}\left(1-\gamma\right)\hat{\pi}^{\omega}_{t+1}.
\end{aligned}
\tag{143}
$$

Define

$$
\Delta\hat{n}^{t}=\frac{1}{1-\alpha_{w}\beta}\left(\hat{n}^{t}-\alpha_{w}\beta\left(E_{t}\hat{n}^{t+1}+E_{t}\hat{\pi}^{\omega}_{t+1}\right)\right). \tag{144}
$$

Then, using (144) and collecting terms in (137), we have

$$
\bar{U}_{f}\left(\hat{U}^{t}_{f}-\alpha_{w}\beta E_{t}\hat{U}^{t+1}_{f}\right)=-\bar{Y}\left(1-\gamma\right)\Delta\hat{n}^{t}+R^{f,t}_{t,t}, \tag{145}
$$

where, using (76) and (61) we have

$$
R^{f,t}_{t,t}=\frac{\sigma}{\sigma-1}\bar{Y}\hat{Y}_{t}-\bar{Y}\left(\hat{w}_{t}-\hat{w}^{*}_{t}+\left(\frac{1}{1-\gamma}\hat{x}_{t}+\hat{Y}^{*}_{t}\right)\right). \tag{146}
$$

Using the solutions for $\widehat{mpl}_t$ and $\widehat{mrs}_t$ from (79) and (87) gives

$$R_{t,t}^{f,t} - R_{t,t}^{f,t*} = \frac{1}{\sigma - 1} \bar{Y} \hat{x}_t - \bar{Y} \left(\hat{w}_t - \widehat{mpl}_t \right) \tag{147}$$

where $R_{t,t}^{f,t*}$ denotes the flexible-price version of $R_{t,t}^{f,t}$.

Second, consider the second expression in (136). We can write this as, using (134)

$$\begin{aligned}
& \nabla_W \bar{U}_u \left(\widehat{\nabla_W U_u^t} - \alpha_w \beta E_t \widehat{\nabla_W U_u^{t+1}} \right) \\
= \;& \overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}_{t,t}^{t} + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\nabla_W \Upsilon} \left(\widehat{\nabla_W \Upsilon}_{t,t+k}^{t} - \widehat{\nabla_W \Upsilon}_{t+1,t+k}^{t} \right) \\
& + \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\nabla_W \Upsilon} \left(\widehat{\nabla_W \Upsilon}_{t+1,t+1+k}^{t} - \widehat{\nabla_W \Upsilon}_{t+1,t+1+k}^{t+1} \right) \\
& + E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^{k-1} (\alpha_w - \alpha_w \alpha) \beta \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \, \overline{\nabla_W \Upsilon} \left(\widehat{\nabla_W \Upsilon}_{t+k,t+k+j}^{t} - \widehat{\nabla_W \Upsilon}_{t+k,t+k+j}^{t+1} \right).
\end{aligned} \tag{148}$$

From (45) we can write

$$1 + \tau_w = \frac{1}{1 + \varepsilon_L} \frac{\sigma - 1}{\bar{u}_C \bar{Y}} \frac{1 - \varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{U}_o \right) + \frac{\varepsilon_L}{1 + \varepsilon_L}. \tag{149}$$

Note that, if τ_w is not chosen as in (45) but is inefficient as in (47), one can easily rewrite the expressions below if one is interested in analyzing the case with distortions. Loglinearizing expression (34), using the above solution for the tax τ_w and that $\bar{u}_C \bar{w} = \bar{v}_L$ gives

$$\begin{aligned}
\overline{\nabla_W \Upsilon} \widehat{\nabla_W \Upsilon}_{t+k,t+k+j}^{t} = \;& \varepsilon_L \left(\bar{u}_C \hat{u}_C \left(C_{t+k+j}, Q_{t+k+j} \right) \bar{w} - \bar{v}_L \hat{v}_L \left(L_{t+k,t+k+j}^{t} (f), Z_{t+k} \right) \right) \frac{\bar{L}}{W(f)} \\
& + \varepsilon_L \left(\bar{u}_C \bar{w} \left(\hat{n}^t - \sum_{l=1}^{k+j} \pi_{t+l}^{\omega} + \hat{w}_{t+k+j} \right) \right) \frac{\bar{L}}{W(f)} \\
& + \frac{\sigma - 1}{\bar{u}_C \bar{Y}} \frac{1 - \varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{U}_o \right) \left(\bar{u}_C \hat{u}_C \left(C_{t+k+j}, Q_{t+k+j} \right) \bar{w} \frac{\bar{L}}{W(f)} \right) \\
& + \frac{\sigma - 1}{\bar{u}_C \bar{Y}} \frac{1 - \varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{U}_o \right) \bar{u}_C \bar{w} \left(\hat{n}^t - \sum_{l=1}^{k+j} \pi_{t+l}^{\omega} + \hat{w}_{t+k+j} + \hat{L}_{t+k,t+k+j}^{t} (f) \right) \frac{\bar{L}}{W(f)},
\end{aligned} \tag{150}$$

where

$$\begin{aligned}
\bar{u}_C \hat{u}_C \left(C_{t+k+j}, Q_{t+k+j} \right) &= \bar{u}_{CC} \bar{C} \hat{C}_{t+k+j} + \bar{u}_{CQ} \bar{Q} \hat{Q}_{t+k+j}, \\
\bar{v}_L \hat{v}_L \left(L_{t+k,t+k+j}^{t} (f), Z_{t+k+j} \right) &= \bar{v}_{LL} \bar{L} \hat{L}_{t+k,t+k+j}^{t} (f) + \bar{v}_{LZ} \bar{Z} \bar{L} \hat{Z}_{t+k+j},
\end{aligned} \tag{151}$$

and

$$\hat{L}^t_{t+k,t+k+j} = -\sigma\left(\hat{q}^t_{t+k} - \sum_{l=k+1}^{k+j}\hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\left(\hat{Y}_{t+k+j} - \hat{A}_{t+k+j}\right) - \gamma\left(\hat{n}^t - \sum_{l=1}^{k+j}\hat{\pi}^{\omega}_{t+l}\right). \tag{152}$$

We then have

$$\begin{aligned}\overline{\nabla_W\Upsilon}\widehat{\nabla_W\Upsilon}^t_{t,t} &= -\varepsilon_L\bar{v}_{LL}\bar{L}\left(-\sigma\hat{q}^t_t(f) - \gamma\hat{n}^t(f)\right)\frac{\bar{L}}{W(f)} + \varepsilon_L\bar{u}_C\bar{w}\frac{\bar{L}}{W(f)}\hat{n}^t \\ &\quad + \frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\bar{u}_C\bar{w}\left((1-\gamma)\hat{n}^t - \sigma\hat{q}^t_t\right)\frac{\bar{L}}{W(f)} + T^{\Delta u,t}_{t,t},\end{aligned} \tag{153}$$

and

$$\begin{aligned}&\overline{\nabla_W\Upsilon}\left(\widehat{\nabla_W\Upsilon}^t_{t,t+k} - \widehat{\nabla_W\Upsilon}^t_{t+1,t+k}\right) \\ =\ &-\varepsilon_L\left(\bar{v}_{LL}\bar{L}\left(\hat{L}^t_{t,t+k}(f) - \hat{L}^t_{t+1,t+k}(f)\right)\right)\frac{\bar{L}}{W(f)} \\ &+\frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\bar{u}_C\bar{w}\left(\hat{L}^t_{t,t+k} - \hat{L}^t_{t+1,t+k}\right)\frac{\bar{L}}{W(f)}, \\ &\overline{\nabla_W\Upsilon}\left(\widehat{\nabla_W\Upsilon}^t_{t+k,t+k+j} - \widehat{\nabla_W\Upsilon}^{t+1}_{t+k,t+k+j}\right) \\ =\ &\varepsilon_L\bar{v}_{LL}\bar{L}\sigma\left(\hat{q}^t_{t+k} - \hat{q}^{t+1}_{t+k}\right)\frac{\bar{L}}{W(f)} \\ &+\varepsilon_L\left(\bar{u}_C\bar{w} + \bar{v}_{LL}\bar{L}\gamma\right)\left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1}\right)\right)\frac{\bar{L}}{W(f)} \\ &+\frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\bar{u}_C\bar{w}\left(1+\varepsilon_L\right)\left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1}\right)\right)\frac{\bar{L}}{W(f)}.\end{aligned} \tag{154}$$

where

$$\begin{aligned}T^{\Delta u,t}_{t,t} &= \varepsilon_L\left(\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{u}_{CQ}\bar{Q}\hat{Q}_t\right)\bar{w}\frac{\bar{L}}{W(f)} - \varepsilon_L\left(\bar{v}_{LL}\bar{L}\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \bar{v}_{LZ}\bar{Z}\bar{L}\hat{Z}_t\right)\frac{\bar{L}}{W(f)} \\ &\quad + \varepsilon_L\bar{u}_C\bar{w}\frac{\bar{L}}{W(f)}\hat{w}_t + \frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\left(\bar{u}_{CC}\bar{C}\hat{C}_t + \bar{u}_{CQ}\bar{Q}\hat{Q}_t\right)\bar{w}\frac{\bar{L}}{W(f)} \\ &\quad + \frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\left(\bar{u}_C\left(\hat{w}_t + \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)\right)\bar{w}\frac{\bar{L}}{W(f)}.\end{aligned} \tag{155}$$

Using expressions (97), (129) and (130) we can write

$$\begin{aligned}\overline{\nabla_W\Upsilon}\widehat{\nabla_W\Upsilon}^t_{t,t} &= \varepsilon_L\bar{v}_{LL}\bar{L}\left(\sigma\hat{q}^t_t(f) + \gamma\hat{n}^t(f)\right)\frac{\bar{L}}{W(f)} + \varepsilon_L\bar{u}_C\bar{w}\frac{\bar{L}}{W(f)}\hat{n}^t + T^{\Delta u,t}_{t,t} \\ &\quad + \frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\bar{u}_C\bar{w}\left((1-\gamma)\hat{n}^t - \sigma\hat{q}^t_t\right)\frac{\bar{L}}{W(f)},\end{aligned} \tag{156}$$

and

$$
\begin{aligned}
&\overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t,t+k} - \widehat{\nabla_W \Upsilon}^t_{t+1,t+k}\right) \\
= \;& \varepsilon_L \bar{v}_{LL} \bar{L} \sigma \left(\hat{q}^t_t - \left(\hat{q}^t_{t+1} + \hat{\pi}_{t+1}\right)\right) \frac{\bar{L}}{W(f)} \\
& - \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{\Upsilon}_o\right) \bar{u}_C \bar{w} \sigma \left(\hat{q}^t_t - \left(\hat{q}^t_{t+1} + \hat{\pi}_{t+1}\right)\right) \frac{\bar{L}}{W(f)}, \\
&\overline{\nabla_W \Upsilon}\left(\widehat{\nabla_W \Upsilon}^t_{t+k,t+k+j} - \widehat{\nabla_W \Upsilon}^{t+1}_{t+k,t+k+j}\right) \\
= \;& \varepsilon_L \left(\bar{u}_C \bar{w} + \bar{v}_{LL} \bar{L} \left(\gamma + \sigma(1-\gamma)\right)\right) \left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^\omega_{t+1}\right)\right) \frac{\bar{L}}{W(f)} \\
& + \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{\Upsilon}_o\right) \bar{u}_C \bar{w} \left(1+\varepsilon_L\right) \left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^\omega_{t+1}\right)\right) \frac{\bar{L}}{W(f)}.
\end{aligned} \tag{157}
$$

Then we can write

$$
\begin{aligned}
&\nabla_W \bar{U}_u \left(\widehat{\nabla_W U^t_u} - \alpha_w \beta E_t \widehat{\nabla_W U^{t+1}_u}\right) \\
= \;& T^{\Delta u,t}_{t,t} + \left(\varepsilon_L \bar{v}_{LL} \bar{L} - \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{U}_o\right) \bar{u}_C \bar{w}\right) \sigma \left(\hat{w}_t - \hat{w}^*_t + \frac{\gamma}{1-\gamma} \hat{x}_t\right) \frac{\bar{L}}{W(f)} \\
& + \frac{1}{1-\alpha_w \beta} E_t \left(\varepsilon_L \left(\bar{u}_C \bar{w} - \bar{v}_{LL} \bar{L} \varepsilon_L\right) \left(\hat{n}^t - \alpha_w \beta \left(\hat{n}^{t+1} + \hat{\pi}^\omega_{t+1}\right)\right) \frac{\bar{L}}{W(f)}\right) \\
& + \frac{1}{1-\alpha_w \beta} E_t \left(\frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{U}_o\right) \bar{u}_C \bar{w} \left(1+\varepsilon_L\right) \left(\hat{n}^t - \alpha_w \beta \left(\hat{n}^{t+1} + \hat{\pi}^\omega_{t+1}\right)\right) \frac{\bar{L}}{W(f)}\right),
\end{aligned} \tag{158}
$$

or, using (144)

$$
\begin{aligned}
&\nabla_W \bar{U}_u \left(\widehat{\nabla_W U^t_u} - \alpha_w \beta E_t \widehat{\nabla_W U^{t+1}_u}\right) \\
= \;& R^{\Delta u,t}_{t,t} + E_t \bar{v}_L \left(\varepsilon_L \left(1 + \varepsilon_L \rho_L\right) + \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{\Upsilon}_o\right) \left(1+\varepsilon_L\right)\right) \frac{\bar{L}}{W(f)} \Delta \hat{n}^t,
\end{aligned} \tag{159}
$$

where

$$
R^{\Delta u,t}_{t,t} = T^{\Delta u,t}_{t,t} + \left(\varepsilon_L \bar{v}_{LL} \bar{L} - \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{\Upsilon}_o\right) \bar{u}_C \bar{w}\right) \sigma \left(\hat{w}_t - \hat{w}^*_t + \frac{\gamma}{1-\gamma} \hat{x}_t\right) \frac{\bar{L}}{W(f)}, \tag{160}
$$

and, using the solutions for $\widehat{mpl}_t$ and $\widehat{mrs}_t$ from (79) and (87)

$$
\begin{aligned}
R^{\Delta u,t}_{t,t} - R^{\Delta u,t*}_{t,t} \;=\;& \bar{v}_L \varepsilon_L \left(-\left(\widehat{mrs}_t - \hat{w}_t\right) - \rho_L \sigma \left(\hat{w}_t - \widehat{mpl}_t\right)\right) \frac{\bar{L}}{W(f)} \\
& + \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{\Upsilon}_o\right) \bar{u}_C \bar{w} \left((1-\sigma)\left(\hat{w}_t - \widehat{mpl}_t\right) + \hat{x}_t\right) \frac{\bar{L}}{W(f)} \\
& + \frac{\sigma-1}{\bar{u}_C \bar{Y}} \frac{1-\varphi}{\varphi} \left(\bar{u} - \bar{v} - \bar{\Upsilon}_o\right) \bar{u}_{CC} \bar{C} \bar{w} \frac{\bar{L}}{W(f)} \hat{x}_t
\end{aligned} \tag{161}
$$

where $R_{t,t}^{\Delta u,t*}$ denotes the flexible-price version of $R_{t,t}^{\Delta u,t}$.

Third, consider the third term in (136). Using expression (132) we have

$$\begin{aligned}
\bar{U}_u\left(\hat{U}_u^t-\alpha_w\beta E_t\hat{U}_u^{t+1}\right) &= \bar{\Upsilon}\hat{\Upsilon}_{t,t}^t+E_t\sum_{k=1}^{\infty}\left(\alpha_w\alpha\beta\right)^k\bar{\Upsilon}\left(\hat{\Upsilon}_{t,t+k}^t-\hat{\Upsilon}_{t+1,t+k}^t\right) \\
&\quad +\alpha_w\beta\sum_{k=0}^{\infty}\left(\alpha_w\alpha\beta\right)^k\bar{\Upsilon}\left(\hat{\Upsilon}_{t+1,t+1+k}^t-\hat{\Upsilon}_{t+1,t+1+k}^{t+1}\right) \qquad (162)\\
&\quad +E_t\sum_{k=2}^{\infty}\left(\alpha_w\beta\right)^{k-1}\left(\alpha_w-\alpha_w\alpha\right)\beta\sum_{j=0}^{\infty}\left(\alpha_w\alpha\beta\right)^j\bar{\Upsilon}\left(\hat{\Upsilon}_{t+k,t+k+j}^t-\hat{\Upsilon}_{t+k,t+k+j}^{t+1}\right),
\end{aligned}$$

where, using (26)

$$\bar{\Upsilon}\hat{\Upsilon}_{t+k,t+k+j}^t=\bar{u}\hat{u}_{t+k+j}-\bar{v}\hat{v}_{t+k+j}, \qquad (163)$$

and where

$$\begin{aligned}
\bar{u}\hat{u}_{t+k+j} &= \bar{u}_C\bar{C}\hat{C}_{t+k+j}+\bar{u}_Q\bar{Q}\hat{Q}_{t+k+j}, \qquad (164)\\
\bar{v}\hat{v}_{t+k+j}^t &= \bar{v}_L\bar{L}\hat{L}_{t+k,t+k+j}^t\left(f\right)+\bar{v}_Z\bar{Z}\hat{Z}_{t+k+j}.
\end{aligned}$$

We have, using (129) and (97)

$$\begin{aligned}
\bar{\Upsilon}\hat{\Upsilon}_{t,t}^t &= \bar{v}_L\bar{L}\left(\sigma\hat{q}_t^t+\gamma\hat{n}^t\right)+\bar{u}_C\bar{C}\hat{C}_t+\bar{u}_Q\bar{Q}\hat{Q}_t-\bar{v}_Z\bar{Z}\hat{Z}_t-\bar{v}_L\bar{L}\frac{1}{1-\gamma}\left(\hat{Y}_t-\hat{A}_t\right), \\
\bar{\Upsilon}\left(\hat{\Upsilon}_{t,t+k}^t-\hat{\Upsilon}_{t+1,t+k}^t\right) &= \bar{v}_L\bar{L}\sigma\left(\hat{q}_t^t-\hat{q}_{t+1}^t-\hat{\pi}_{t+1}\right), \qquad (165)\\
\bar{\Upsilon}\left(\hat{\Upsilon}_{t+k,t+k+j}^t-\hat{\Upsilon}_{t+k,t+k+j}^{t+1}\right) &= \bar{v}_L\bar{L}\left(\sigma\left(1-\gamma\right)+\gamma\right)\left(\hat{n}^t-\hat{n}^{t+1}-\hat{\pi}_{t+1}^{\omega}\right).
\end{aligned}$$

Then, using expressions (130) (144), we have

$$\bar{U}_u\left(\hat{U}_u^t-\alpha_w\beta E_t\hat{U}_u^{t+1}\right)=\bar{v}_L\bar{L}\left(\sigma\left(1-\gamma\right)+\gamma\right)\Delta\hat{n}^t+R_{t,t}^{u,t}, \qquad (166)$$

where

$$R_{t,t}^{u,t}=\bar{u}_C\bar{C}\hat{C}_t+\bar{u}_Q\bar{Q}\hat{Q}_t-\bar{v}_Z\bar{Z}\hat{Z}_t-\bar{v}_L\bar{L}\frac{1}{1-\gamma}\left(\hat{Y}_t-\hat{A}_t\right)+\bar{v}_L\bar{L}\sigma\left(\hat{w}_t-\hat{w}_t^*+\frac{\gamma}{1-\gamma}\hat{x}_t\right). \qquad (167)$$

Using the solutions for $\widehat{mpl}_t$ and $\widehat{mrs}_t$ from (79) and (87), gives, using that we from (49) have $\bar{v}_L\bar{L}=\bar{u}_C\bar{C}\left(1-\gamma\right)$

$$R_{t,t}^{u,t}-R_{t,t}^{u,t*}=\bar{u}_C\bar{C}\hat{x}_t-\bar{v}_L\bar{L}\frac{1}{1-\gamma}\hat{x}_t+\bar{v}_L\bar{L}\sigma\left(\hat{w}_t-\widehat{mpl}_t\right)=\bar{v}_L\bar{L}\sigma\left(\hat{w}_t-\widehat{mpl}_t\right) \qquad (168)$$

where $R^{u,t*}_{t,t}$ denotes the flexible-price version of $R^{u,t}_{t,t}$.

Finally, to rewrite the last term in expression (136) we use (135) and hence we have

$$
\begin{aligned}
&\nabla_W \bar{U}_f \left(-\widehat{\nabla_W U_f^t} - \alpha_w \beta E_t \left(-\widehat{\nabla_W U_f^{t+1}}\right)\right) \\
= \;& \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t,t}\right) + E_t \sum_{k=1}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t,t+k} - \left(-\widehat{\nabla_W \phi}^t_{t+1,t+k}\right)\right) \qquad (169)\\
& + \alpha_w \beta \sum_{k=0}^{\infty} (\alpha_w \alpha \beta)^k \, \overline{\frac{\nabla_W \phi}{\partial W(f)}}\left(-\widehat{\nabla_W \phi}^t_{t+1,t+1+k} - \left(-\widehat{\nabla_W \phi}^{t+1}_{t+1,t+1+k}\right)\right) \\
& + E_t \sum_{k=2}^{\infty} (\alpha_w \beta)^{k-1} (\alpha_w - \alpha_w \alpha)\, \beta \sum_{j=0}^{\infty} (\alpha_w \alpha \beta)^j \, \overline{\nabla_W \phi}\left(-\widehat{\nabla_W \phi}^t_{t+k,t+k+j} - \left(-\widehat{\nabla_W \phi}^{t+1}_{t+k,t+k+j}\right)\right),
\end{aligned}
$$

where

$$
\begin{aligned}
-\widehat{\nabla_W \phi}^t_{t+k,t+k+j} \;=\; & -\sigma \left(\hat{q}^t_{t+k} - \sum_{l=k+1}^{k+j} \hat{\pi}_{t+l}\right) + \frac{1}{1-\gamma}\left(\hat{Y}_{t+k+j} - \hat{A}_{t+k+j}\right) \qquad (170)\\
& + (1-\gamma)\left(\hat{n}^t - \sum_{l=1}^{k+j} \hat{\pi}^{\omega}_{t+l}\right) + \hat{w}_{t+k+j},
\end{aligned}
$$

and, using that $\overline{tc} = \bar{Y}$, we get

$$
\overline{\nabla_W \phi} = -(1-\gamma)\frac{\bar{Y}}{W(f)}. \qquad (171)
$$

Then, from expression (100)

$$
\begin{aligned}
-\widehat{\nabla_W \phi}^t_{t,t} &= -\sigma \hat{q}^t_t + \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + (1-\gamma)\hat{n}^t + \hat{w}_t, \\
-\widehat{\nabla_W \phi}^t_{t,t+k} - \left(-\widehat{\nabla_W \phi}^t_{t+1,t+k}\right) &= -\sigma\left(\hat{q}^t_t - \left(\hat{q}^t_{t+1} + \hat{\pi}_{t+1}\right)\right), \qquad (172)\\
-\widehat{\nabla_W \phi}^t_{t+k,t+k+j} - \left(-\widehat{\nabla_W \phi}^{t+1}_{t+k,t+k+j}\right) &= -\sigma\left(\hat{q}^t_{t+k} - \hat{q}^{t+1}_{t+k}\right) + (1-\gamma)\left(\hat{n}^t - \left(\hat{n}^{t+1} + \hat{\pi}^{\omega}_{t+1}\right)\right).
\end{aligned}
$$

Using a similar argument as above gives, using (129), (130) and (144)

$$
\nabla_W \bar{U}_f \left(-\widehat{\nabla_W U_f^t} - \alpha_w \beta E_t \left(-\widehat{\nabla_W U_f^{t+1}}\right)\right) = R^{\Delta f,t}_{t,t} - (1-\gamma)\frac{\bar{Y}}{W(f)}(1-\gamma)(1-\sigma)\Delta \hat{n}^t, \qquad (173)
$$

where

$$
R^{\Delta f,t}_{t,t} = -(1-\gamma)\frac{\bar{Y}}{W(f)}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) + \hat{w}_t - \sigma\left(\hat{w}_t - \hat{w}^*_t + \frac{\gamma}{1-\gamma}\hat{x}_t\right)\right). \qquad (174)
$$

Using the solutions for $\widehat{mpl}_t$ and $\widehat{mrs}_t$ from (79) and (87), gives

$$
R^{\Delta f,t}_{t,t} - R^{\Delta f,t*}_{t,t} = -(1-\gamma)\frac{\bar{Y}}{W(f)}\left(\hat{x}_t + (1-\sigma)\left(\hat{w}_t - \widehat{mpl}_t\right)\right) \qquad (175)
$$

where $R_{t,t}^{\Delta f,t*}$ denotes the flexible-price version of $R_{t,t}^{\Delta f,t}$.

We can write the third term (145) in the first-order condition (when subtracting the flexible-price first-order condition) as

$$\frac{1}{\bar{U}_f}\bar{U}_f\left(\hat{U}_f^t-\alpha_w\beta E_t\hat{U}_f^{t+1}\right)=\frac{1}{1-\alpha_w\beta}\frac{\bar{v}_L\bar{L}}{W\left(f\right)}\bar{Y}\left(-\left(1-\gamma\right)\Delta\hat{n}^t+\frac{1}{\sigma-1}\hat{x}_t-\left(\hat{w}_t-\widehat{mpl}_t\right)\right), \tag{176}$$

and the last term (173)

$$\begin{aligned}&\nabla_W\bar{U}_f\left(-\widehat{\nabla_W U_f^t}-\alpha_w\beta E_t\left(-\widehat{\nabla_W U_f^{t+1}}\right)\right) \qquad (177)\\ =\;&\frac{1}{1-\alpha_w\beta}\frac{\bar{v}_L\bar{L}}{W\left(f\right)}\bar{Y}\left(-\frac{1}{\sigma-1}\hat{x}_t+\left(\hat{w}_t-\widehat{mpl}_t\right)+\left(1-\gamma\right)\Delta\hat{n}^t\right).\end{aligned}$$

These two terms eliminates each other in expression (136). The wage setting "Phillips" curve can then be derived from (159) and (166).

Using expressions (145), (159), (166) and (173), the definitions of $R_{t,t}^{u,t}$, $R_{t,t}^{\Delta u,t}$, $R_{t,t}^{f,t}$ and $R_{t,t}^{\Delta f,t}$, the steady state values of $\bar{U}_f$, $\nabla_W\bar{U}_u$, $\bar{U}_u$ and $\nabla_W\bar{U}_f$ and that the only difference between the flexible and sticky price values of these terms is that variables are evaluated at their flexible and sticky price levels, respectively, the first-order condition (136) can be rewritten as

$$\Phi_d\Delta\hat{n}^t+\Phi_x\hat{x}_t+\Phi_w\left(\hat{w}_t-\hat{w}_t^*\right)=0, \tag{178}$$

where, using the solution for labor taxes gives and the definition of ε_L, given in expression (14), we can write

$$\begin{aligned}\Phi_d &= \varepsilon_L\frac{1}{\sigma-1}\left(\varphi\left(2+\left(1+\rho_L\right)\varepsilon_L\right)-\left(1+\varepsilon_L\right)\right)\bar{v}_L\bar{L}-\left(1-\varphi\right)\left(\sigma-1\right)\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\left(1-\gamma\right)^2\\ \Phi_x &= \left(-\varphi\frac{\varepsilon_L}{\sigma-1}\left(\rho_C-\rho_L\frac{1-\sigma\gamma}{1-\gamma}\right)-\left(1-\varphi\right)\sigma\gamma\right)\bar{v}_L\bar{L} \qquad (179)\\ &\quad+\left(1-\varphi\right)\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\left(1-\sigma\gamma-\rho_C\left(1-\gamma\right)\right)\\ \Phi_w &= -\left(-\varphi\frac{\varepsilon_L}{\sigma-1}\left(1-\sigma\rho_L\right)+\left(1-\varphi\right)\left(1-\gamma\right)\sigma\right)\bar{v}_L\bar{L}+\left(1-\varphi\right)\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)\left(1+\varepsilon_L\right)\end{aligned}$$

There is a potential problem associated with adjusting τ_w as in Erceg, Henderson, and Levin (2000) in order to achieve efficiency, since this approach leads to inconsistencies when $\varphi<1$ (see the section on comparing our model with the Erceg, Henderson, and Levin (2000) model in Carlsson and Westermark (2006)). It is of course possible to use both τ_w and $\bar{\Upsilon}_o$ to eliminate distortions on the labor market. Then there is a continuum of possible ways to ensure efficiency, all leading to different wage setting behavior. The way we pin down a unique wage setting curve, is to adjust both τ_w and

$\bar{\Upsilon}_o$ such that each party receives a share of the total surplus corresponding to their bargaining power.[7]

To achieve this, we do as follows. First, we know that the share of the surplus that accrues to the firm in steady state is $1-\varphi$. We also know that the firm always must get $\tau\bar{Y}$ in steady state. Then we have to adjust $\bar{\Upsilon}_o$ so that the firm share $1-\varphi$ is always $\tau\bar{Y}$. We then need to derive total surplus. To analyze total surplus, we need to transform it into consumption terms. This is done by multiplying the steady state payoff for the worker by $\frac{1}{\bar{u}_C}$ implying that total surplus is

$$\frac{\bar{u}-\bar{v}-\bar{\Upsilon}_o}{\bar{u}_C}+\tau\bar{Y}. \tag{180}$$

The firm then gets

$$\tau\bar{Y}=(1-\varphi)\left(\frac{\bar{u}-\bar{v}-\bar{\Upsilon}_o}{\bar{u}_C}+\tau\bar{Y}\right). \tag{181}$$

Using that $\bar{u}_C(1-\gamma)\bar{C}=\bar{v}_L\bar{L}$, this expression can be rewritten as

$$\bar{u}-\bar{v}-\bar{\Upsilon}_o=\frac{\varphi}{1-\varphi}\tau\bar{u}_C\bar{C}=\frac{\varphi}{1-\varphi}\frac{1}{\sigma-1}\frac{\bar{v}_L\bar{L}}{(1-\gamma)}. \tag{182}$$

Note that, when using the solution for $\bar{u}-\bar{v}-\bar{\Upsilon}_o$ and the solution for τ from (38) in the labor tax rate τ_w from (45) we get

$$\begin{aligned}\tau_w &= \frac{1}{(1-\sigma)(1-\gamma)}\left(\frac{\sigma-1}{\bar{u}_C\bar{Y}}\frac{1-\varphi}{\varphi}\left(\bar{u}-\bar{v}-\bar{\Upsilon}_o\right)-(\gamma+\sigma(1-\gamma))\right)-1 \\ &= \frac{1}{(1-\sigma)(1-\gamma)}(1-(\gamma+\sigma(1-\gamma)))-1=0.\end{aligned} \tag{183}$$

Thus, the method above implies that we only adjust $\bar{\Upsilon}_o$ to achieve efficiency. We then eliminate the two distortions in the economy stemming from monopoly power in the intermediate goods market and from union bargaining power in the labor market by using τ and $\bar{\Upsilon}_o$. Using (182) in (179) gives

$$\begin{aligned}\Phi_d &= \left(\varepsilon_L\frac{1}{\sigma-1}(\varphi(2+(1+\rho_L)\varepsilon_L)-(1+\varepsilon_L))-\varphi(1-\gamma)\right)\bar{v}_L\bar{L} \\ \Phi_x &= \left(\left(-\varphi\frac{\varepsilon_L}{\sigma-1}\left(\rho_C-\rho_L\frac{1-\sigma\gamma}{1-\gamma}\right)-(1-\varphi)\sigma\gamma\right)+\varphi\frac{1-\sigma\gamma-\rho_C(1-\gamma)}{(\sigma-1)(1-\gamma)}\right)\bar{v}_L\bar{L} \\ \Phi_w &= \left(-\left(-\varphi\frac{\varepsilon_L}{\sigma-1}(1-\sigma\rho_L)+(1-\varphi)(1-\gamma)\sigma\right)+\varphi\frac{1+\varepsilon_L}{(\sigma-1)(1-\gamma)}\right)\bar{v}_L\bar{L}\end{aligned} \tag{184}$$

[7] This is how the surplus is divided in linear bargaining problems, see Osborne and Rubinstein (1990).

Using (184) we get, when dividing expression (178) with Φ_d and dividing through with $1+\varepsilon_L$ gives

$$\begin{aligned}
\frac{\Phi_x}{\Phi_d} &= \frac{\varphi\left(\frac{1-\sigma\gamma}{1-\gamma}\frac{1+\rho_L\varepsilon_L}{1+\varepsilon_L}-\rho_C\right)+(1-\varphi)\,\sigma\frac{\gamma}{1-\gamma}}{\varphi\left(\frac{\varepsilon_L}{1+\varepsilon_L}(1+\rho_L\varepsilon_L)+1\right)-(1-\varphi)\,\varepsilon_L}, \\
\frac{\Phi_w}{\Phi_d} &= \frac{\varphi\left(\frac{\varepsilon_L}{1+\varepsilon_L}(1-\sigma\rho_L)+\frac{1}{1-\gamma}\right)+(1-\varphi)\,\sigma}{\varphi\left(\frac{\varepsilon_L}{1+\varepsilon_L}(1+\rho_L\varepsilon_L)+1\right)-(1-\varphi)\,\varepsilon_L}.
\end{aligned} \tag{185}$$

To express the wage setting equation in terms of wage inflation, we need to express relative wages in terms of wage inflation. The wage evolution equation is, recalling that $W_t^o(f)$ is the optimal wage for firm f when renegotiating wages in period t

$$W_t = \alpha_w \int_0^1 \bar{\pi} W_{t-1}(f)\,df + (1-\alpha_w)\int_0^1 W_t^o(f)\,df. \tag{186}$$

Using that $\frac{W_{t-1}}{W_t} = \frac{1}{\pi_t^\omega}$ gives

$$1 = \alpha_w \bar{\pi}\frac{1}{\pi_t^\omega} + (1-\alpha_w)\int n^t(f)\,df. \tag{187}$$

Letting $n^t = \int n^t(f)\,df$ and loglinearizing gives

$$\hat{n}^t = \frac{\alpha_w}{1-\alpha_w}\hat{\pi}_t^\omega. \tag{188}$$

Using expression (188) in (144) yields

$$\Delta\hat{n}^t = \frac{1}{1-\alpha_w\beta}\frac{\alpha_w}{1-\alpha_w}\left(\hat{\pi}_t^\omega - \beta E_t\hat{\pi}_{t+1}^\omega\right), \tag{189}$$

and letting

$$\Pi_1 = (1-\alpha_w\beta)\frac{1-\alpha_w}{\alpha_w}, \tag{190}$$

the wage setting Phillips curve is

$$\hat{\pi}_t^\omega = \beta E_t\hat{\pi}_{t+1}^\omega - \Pi_1\left(\frac{\Phi_x}{\Phi_d}\hat{x}_t + \frac{\Phi_w}{\Phi_d}(\hat{w}_t - \hat{w}_t^*)\right). \tag{191}$$

Hence, the first-order condition for wage setting is the wage setting Phillips curve; see Carlsson and Westermark (2006) for an in-depth discussion on the intuition for (191) as well as for (192) below.

Using the solutions for $\widehat{mpl}_t$ and $\widehat{mrs}_t$ from (79) and (87), (159) and (166), together with (161) and (168), expression (178) can be rewritten as, where $\kappa = \varphi\left(\frac{\varepsilon_L}{1+\varepsilon_L}(1+\rho_L\varepsilon_L)+1\right) - (1-\varphi)\varepsilon_L$

$$\begin{aligned} \hat{\pi}_t^{\omega} &= \beta E_t \hat{\pi}_{t+1}^{\omega} + (1-\varphi)\frac{\Pi_1}{\kappa}\sigma\left(\hat{w}_t - \widehat{mpl}_t\right) \\ &\quad + \frac{\Pi_1}{\kappa}\varphi\left((\hat{w}_t - \widehat{mrs}_t) - (\rho_L\sigma + (1-\sigma))\left(\hat{w}_t - \widehat{mpl}_t\right) + (\rho_C - 1)\hat{x}_t\right). \end{aligned} \tag{192}$$

4.5 Real wage evolution

The real wage today can be written as a function of the previous period real wage as follows

$$\frac{W_t}{P_t} = \frac{\pi_t^{\omega} W_{t-1}}{\pi_t P_{t-1}}. \tag{193}$$

Log-linearizing gives

$$\hat{w}_t = \hat{w}_{t-1} + \hat{\pi}_t^{\omega} - \hat{\pi}_t. \tag{194}$$

5 Welfare

When computing welfare in this model, a second-order approximation in logs is used, resulting in that we can relate welfare to the variance in relative prices and wages. Also, the output gap matters because it distorts the economywide relationship between consumption and leisure. Before analyzing welfare, we first compute second-order approximations of L_t and Y_t, the relationship between real variation and price variation and finally persistence in price variability.

5.1 Quadratic approximation of L_t and Y_t

We first proceed by looking at a quadratic approximation of L_t and Y_t. Aggregate demand of labor by firms is, where the integral is taken over firms

$$L_t = \int_0^1 L(f)\, df. \tag{195}$$

Then a quadratic approximation is

$$\hat{L}_t = E_f \hat{L}_t(f) + \frac{1}{2} var_f \hat{L}_t(f). \tag{196}$$

Using the definition of the composite good in (1), we can similarly derive

$$E_f \hat{Y}_t(f) = \hat{Y}_t - \frac{1}{2}\frac{\sigma - 1}{\sigma} var_f \hat{Y}_t(f). \tag{197}$$

Now, let us express (196) in terms of aggregate variables and variances. Taking the expectation of (92), using $\hat{w}_t = E_f \hat{w}_t(f)$ and (76) gives

$$E_f \hat{L}_t(f) = E_f \hat{Y}_t(f) - \frac{1}{1-\gamma}\hat{A}_t + \frac{\gamma}{1-\gamma}\hat{Y}_t. \tag{198}$$

Then, using (197) in (198) and expression (196) we get

$$\hat{L}_t = E_f \hat{L}_t(f) + \frac{1}{2} var_f \hat{L}_t(f) = \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) - \frac{1}{2}\frac{\sigma-1}{\sigma} var_f \hat{Y}_t(f) + \frac{1}{2} var_f \hat{L}_t(f). \tag{199}$$

5.2 Relationship between real and price variability

In this section, we relate price variability to variability in real variables, which, in turn, creates a link between price dispersion and welfare. We start by computing $var_f \hat{L}_t(f)$ as a function of $var_f \hat{P}_t(f)$ and $var_f \hat{w}_t(f)$. We also use that $w_t(f) = \frac{W_t(f)}{P_t} = n_t(f) w_t$ from (88). First, note that it follows that $var_f \hat{w}_t(f) = var_f \hat{n}_t(f)$. Second, let us find $var_f \hat{L}_t(f)$. Since

$$\hat{L}_t(f) = -\sigma \hat{q}_t(f) + \frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right) - \gamma\left(\hat{n}_t(f) + \hat{w}_t\right), \tag{200}$$

we must have

$$var_f \hat{L}_t(f) = \sigma^2 var_f \hat{q}_t(f) + \gamma^2 var_f \hat{n}_t(f) + 2\sigma\gamma cov_f \hat{q}_t(f)\hat{n}_t(f). \tag{201}$$

Third, let us find $cov_f \hat{L}_t(f)\hat{w}_t(f)$. From (200) we have

$$cov_f \hat{L}_t(f)\hat{w}_t(f) = -\gamma var_f \hat{n}_t(f) - \sigma cov_f \hat{q}_t(f)\hat{n}_t(f). \tag{202}$$

Finally, we need to find $cov_f \hat{q}_t(f)\hat{n}_t(f)$. To do this, note that prices depend on wages. The price setting relation (114) becomes, using $\widehat{mpl}_t(f) = \hat{A}_t + \gamma\hat{w}_t(f) - \gamma\hat{p}^c_t$, assuming that prices last were changed in $t-j$ and letting $\hat{n}_{t-j}(f)$ denote the relevant relative wage

$$\begin{aligned} 0 &= \frac{1}{1-\alpha_w\alpha\beta}\hat{q}_{t-j}(f) \\ &\quad + E_{t-j}\sum_{k=0}^{\infty}(\alpha_w\alpha\beta)^k\left[-\sum_{l=1}^{k}\hat{\pi}_{t-j+l} - (1-\gamma)\left(\hat{n}_{t-j}(f) - \sum_{l=1}^{k}\hat{\pi}^w_{t-j+l} + \hat{w}_{t-j+k}\right) + \hat{A}_{t-j+k} - \gamma\hat{p}^c_{t-j+k}\right]. \end{aligned} \tag{203}$$

Setting the aggregate variables equal to some constant Ξ, we can write[8]

$$\hat{q}_{t-j}(f) = (1-\gamma)\hat{n}_{t-j}(f) + \Xi, \tag{204}$$

[8] We can ignore the aggregate variables, since we are only interested in variances and covariances across firms

where Ξ is independent of f. Note that the relative price in period j is

$$q_t(f) = \bar{\pi}^j q_{t-j}(f) \frac{P_{t-j}}{P_t}, \tag{205}$$

and hence

$$\hat{q}_t(f) = \hat{q}_{t-j}(f) - \sum_{l=1}^{j} \hat{\pi}_{t-l} \iff \hat{q}_{t-j}(f) = \hat{q}_t(f) + \sum_{l=1}^{j} \hat{\pi}_{t-l}. \tag{206}$$

A similar argument establishes that

$$\hat{n}_{t-j}(f) = \hat{n}_t(f) + \sum_{l=1}^{j} \hat{\pi}_{t-l}^{\omega}, \tag{207}$$

and hence we can write (204) as

$$\hat{q}_t(f) = (1-\gamma)\hat{n}_t(f) + \Xi', \tag{208}$$

where Ξ' is independent of f. Then we have

$$cov \hat{q}_t(f) \hat{n}_t(f) = (1-\gamma) var_f \hat{n}_t(f), \tag{209}$$

and hence (201) can be rewritten as, using that $var_f \hat{q}_t(f) = var_f \hat{P}_t(f)$ and $var_f \hat{n}_t(f) = var_f \hat{w}_t(f)$

$$var_f \hat{L}_t(f) = \sigma^2 var_f \hat{P}_t(f) + \left(\gamma^2 + 2\sigma\gamma(1-\gamma)\right) var_f \hat{w}_t(f), \tag{210}$$

and, taking a quadratic approximation of (3)

$$var_f \hat{Y}_t(f) = \sigma^2 var_f \hat{P}_t(f). \tag{211}$$

5.3 Variance Persistence

Since prices and wages are not fully flexible, the variance of the price and wage distribution across firms are persistent. We want to find the variance of the distributions today as function of previous variances and inflation. To do this, let us express $var_f(\log P_t(f))$ and $var_f(\log W_t(f))$ in terms of squared inflation and wage inflation. Combining this with (210) and (211) we get a relationship between real variability and inflation, which enables us to write welfare in terms of inflation and wage inflation. Let $\bar{P}_t = E_f \log P_t(f)$. We have

$$var_f(\log P_t(f)) = E_f\left(\log P_t(f) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta \bar{P}_t\right)^2, \tag{212}$$

where

$$\Delta\bar{P}_t = \bar{P}_t - \log\bar{\pi} - \bar{P}_{t-1}. \tag{213}$$

Let us rewrite $\Delta\bar{P}_t$ in terms of inflation. Since $\log P_t = E_f \log P_t(f) = \bar{P}_t$ we can rewrite $\Delta\bar{P}_t$ as[9]

$$\Delta\bar{P}_t = \log P_t - \log\bar{\pi} - \log P_{t-1} = \log\pi_t - \log\bar{\pi} = \hat{\pi}_t. \tag{214}$$

We also define

$$\Delta\bar{W}_t = \bar{W}_t - \log\bar{\pi} - \bar{W}_{t-1} = (1-\alpha_w)\left(\log W_t^o - \log\bar{\pi} - \bar{W}_{t-1}\right). \tag{215}$$

Similarly, let us rewrite $\Delta\bar{W}_t$ in terms of wage inflation. Note that we have $\log W_t = \bar{W}_t$.[10] Then $\Delta\bar{W}_t$ can be rewritten as

$$\Delta\bar{W}_t = \log W_t - \log\bar{\pi} - \log W_{t-1} = \log\pi_t^w - \log\bar{\pi} = \hat{\pi}_t^w. \tag{216}$$

We can write the variance in (212) as, using that when wages are changed, they are the same for all firms, i.e., $W_t^o(f) = W_t^o$ for all f

$$\begin{aligned} &var_f(\log P_t(f)) = \alpha_w\alpha E_f\left(\log\bar{\pi}P_{t-1}(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2 \\ &+(1-\alpha)\,\alpha_w E_f\left(\log P_t^o(W_t(f)) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 \\ &+(1-\alpha_w)\left(\log P_t^o(W_t^o) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2. \end{aligned} \tag{217}$$

We now rewrite expression (217) in terms of lagged variance in prices, variance in wages and inflation and wage inflation. To do this, we need to rewrite the third and fourth term in expression (217). To rewrite the third term, let us express $E_f\left(\log P_t^o(W_t(f)) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2$ in terms of $\Delta\bar{P}_t$

[9] Note that we are interested in the first moment. Then this equality follows from using a first-order Taylor approximation of the price level $P_t^{1-\sigma} = \int_0^1 P_t(f)^{1-\sigma}\,df$

$$\bar{P}_t + (1-\sigma)\,\bar{P}_t^{-\sigma}\left(\bar{P}_t\hat{P}_t - \bar{P}_t\right) = \int_0^1\left(\bar{P}_t(f) + (1-\sigma)\,\bar{P}_t(f)^{-\sigma}\left(\bar{P}_t(f)\,\hat{P}_t(f) - \bar{P}_t(f)\right)\right)df.$$

Since $\bar{P}_t = \bar{P}_t(f)$ we have

$$\hat{P}_t = \int_0^1 \hat{P}_t(f)\,df = E_f\hat{P}_t,$$

or, by the definition of $\hat{P}_t$ and $\hat{P}_t(f)$ (they are constructed as log-deviations around the same mean).

[10] This follows from a similar argument as in the previous footnote.

and $\Delta \bar{W}_t$. We have

$$\Delta \bar{P}_t = (1-\alpha)\,\alpha_w \left(E_f \log P_t^o\left(W_t(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right) + (1-\alpha_w)\left(E_f \log P_t^o\left(W_t^o\right) - \log \bar{\pi} - \bar{P}_{t-1}\right). \tag{218}$$

Using a first order approximation of $\log P_t^o(W_t^o)$ gives, using that, from (208) we have $\frac{\partial \log P_t^o(\bar{\pi} W_{t-1}(f))}{\partial \log W_t} = (1-\gamma)$ [11]

$$\begin{aligned}
\log P_t^o\left(W_t^o\right) &\approx \log P_t^o\left(\bar{\pi} W_{t-1}(f)\right) + \frac{\partial \log P_t^o\left(\bar{\pi} W_{t-1}(f)\right)}{\partial \log W_t}\left(\log W_t^o - \log \bar{\pi} W_{t-1}(f)\right) \qquad (219)\\
&= \log P_t^o\left(\bar{\pi} W_{t-1}(f)\right) + (1-\gamma)\left(\log W_t^o - \log \bar{\pi} W_{t-1}(f)\right).
\end{aligned}$$

Using that we have $W_t(f) = \bar{\pi} W_{t-1}(f)$ for firms that do not change prices and since $E_f\left(\log \bar{\pi} W_{t-1}(f)\right) = \log \bar{\pi} + \bar{W}_{t-1}$ and using (215) we have

$$\Delta \bar{P}_t - (1-\gamma)\,\Delta \bar{W}_t = \left((1-\alpha)\,\alpha_w + (1-\alpha_w)\right)\left(E_f \log P_t^o\left(W_t(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right). \tag{220}$$

Also, we have

$$\begin{aligned}
& E_f\left(\log P_t^o\left(W_t(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 \qquad (221)\\
=\;& E_f\left(\log P_t^o\left(W_t(f)\right) - E_f \log P_t^o\left(W_t(f)\right) + E_f \log P_t^o\left(W_t(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2\\
=\;& var_f \log P_t^o\left(W_t(f)\right) + \left(E_f \log P_t^o\left(W_t(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2,
\end{aligned}$$

and, using (220) and (221) we get

$$E_f\left(\log P_t^o\left(W_t(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 = var_f \log P_t^o\left(W_t(f)\right) + \frac{\left(\Delta \bar{P}_t - (1-\gamma)\,\Delta \bar{W}_t\right)^2}{\left((1-\alpha)\,\alpha_w + (1-\alpha_w)\right)^2}. \tag{222}$$

To rewrite the fourth term in (217), using that $\log P_t^o(W_t^o)$ is the same for all firms that change wages and the loglinearization of $\log P_t^o(W_t^o)$ (i.e. (219)) we can write

$$\begin{aligned}
& \left(\log P_t^o\left(W_t^o\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 \qquad (223)\\
=\;& E_f\left(\log P_t^o\left(\bar{\pi} W_{t-1}(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 + E_f\left((1-\gamma)\left(\log W_t^o - \log \bar{\pi} W_{t-1}(f)\right)\right)^2\\
& + 2(1-\gamma)\, E_f\left(\log p_t^o\left(\bar{\pi} W_{t-1}(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)\left(\log W_t^o - \log \bar{\pi} W_{t-1}(f)\right).
\end{aligned}$$

The three terms in expression (223) can be written as, using (221) and that we have $W_t(f) = \bar{\pi} W_{t-1}(f)$

[11] Note that we linearize around the previous wage for each firm here. This enables us to express the second term in the $\Delta \bar{P}_t$ expression in terms of $\Delta \bar{W}_t$ and the first term. Also, it is one of the two obvious point around to loglinearize. One could alternatively choose the steady state wage.

for firms that do not change wages

$$
\begin{aligned}
& E_f \left(\log P_t^o \left(\bar{\pi} W_{t-1}(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)^2 \qquad (224) \\
= \; & var_f \log P_t^o \left(W_t(f)\right) + \frac{1}{\left(\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)\right)^2} \left(\Delta \bar{P}_t - \left(1-\gamma\right)\Delta \bar{W}_t\right)^2,
\end{aligned}
$$

using (215)

$$
E_f \left(\left(1-\gamma\right)\left(\log W_t^o - \log \bar{\pi} W_{t-1}(f)\right)\right)^2 = \left(1-\gamma\right)^2 \left(\frac{1}{\left(1-\alpha_w\right)^2}\left(\Delta \bar{W}_t\right)^2 + var_f \log W_{t-1}(f)\right), \qquad (225)
$$

and, using (215), (219) and (220) that

$$
\begin{aligned}
& E_f \left(\log P_t^o \left(\bar{\pi} W_{t-1}(f)\right) - \log \bar{\pi} - \bar{P}_{t-1}\right)\left(\log W_t^o - \log \bar{\pi} W_{t-1}(f)\right) \qquad (226) \\
= \; & \frac{1}{\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)} \left(\Delta \bar{P}_t - \left(1-\gamma\right)\Delta \bar{W}_t\right) \frac{1}{1-\alpha_w}\Delta \bar{W}_t - \left(1-\gamma\right) var_f \log W_{t-1}(f).
\end{aligned}
$$

Using expressions (224), (225) and (226) in (223) gives the fourth term in (217) as

$$
\begin{aligned}
& var_f \log P_t^o \left(W_t(f)\right) + \frac{1}{\left(\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)\right)^2}\left(\Delta \bar{P}_t - \left(1-\gamma\right)\Delta \bar{W}_t\right)^2 \\
& + \left(1-\gamma\right)^2 \left(\frac{1}{\left(1-\alpha_w\right)^2}\left(\Delta \bar{W}_t\right)^2 + var_f \log W_{t-1}(f)\right) \qquad (227) \\
& + 2\left(1-\gamma\right)\left(\frac{\left(\Delta \bar{P}_t - \left(1-\gamma\right)\Delta \bar{W}_t\right)\Delta \bar{W}_t}{\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)}\frac{1}{1-\alpha_w} - \left(1-\gamma\right) var_f \log W_{t-1}(f)\right).
\end{aligned}
$$

Let us now collect the arguments above to rewrite expression (217) in terms of lagged variance in prices, variance in wages and inflation and wage inflation. The expression $var_f \log P_t^o \left(W_t(f)\right)$ involves firms that do not change prices. From (208) we then have

$$
var_f \log P_t^o \left(W_t(f)\right) = \left(1-\gamma\right)^2 var_f \log W_t(f) = \left(1-\gamma\right)^2 var_f \log W_{t-1}(f). \qquad (228)
$$

Then we have, using (222), (227) and (228) in (217)

$$
\begin{aligned}
var_f \left(\log P_t(f)\right) \; = \; & \alpha_w \alpha var_f \left(\log P_{t-1}(f)\right) + \left(1-\alpha\right)\alpha_w \left(1-\gamma\right)^2 var_f \log W_{t-1}(f) \qquad (229) \\
& + \frac{\alpha_w}{\left(1-\alpha\right)\alpha_w + \left(1-\alpha_w\right)}\left(\alpha\left(\Delta \bar{P}_t\right)^2 + \frac{1-\alpha}{1-\alpha_w}\left(1-\gamma\right)^2\left(\Delta \bar{W}_t\right)^2\right).
\end{aligned}
$$

Using expressions (214) and (216) gives

$$
\begin{aligned}
var_f\left(\log P_t\left(f\right)\right) &= \alpha_w \alpha var_f\left(\log P_{t-1}\left(f\right)\right)+\left(1-\alpha\right)\alpha_w\left(1-\gamma\right)^2 var_f \log W_{t-1}\left(f\right) \\
&+\frac{\alpha_w}{\left(1-\alpha\right)\alpha_w+\left(1-\alpha_w\right)}\left(\alpha\left(\hat{\pi}_t\right)^2+\frac{1-\alpha}{1-\alpha_w}\left(1-\gamma\right)^2\left(\hat{\pi}_t^w\right)^2\right).
\end{aligned}
\tag{230}
$$

For wages, we can write, using a similar method as in (212), and using (215) we have

$$
var_f\left(\log W_t\left(f\right)\right)=\alpha_w var_f\left(\log W_{t-1}\left(f\right)\right)+\frac{\alpha_w}{1-\alpha_w}\left(\Delta\bar{W}_t\right)^2. \tag{231}
$$

Using expression (216) this gives

$$
var_f\left(\log W_t\left(f\right)\right)=\alpha_w var_f\left(\log W_{t-1}\left(f\right)\right)+\frac{\alpha_w}{1-\alpha_w}\left(\hat{\pi}_t^w\right)^2+o\left(\|\xi\|^3\right), \tag{232}
$$

where $o\left(\|\xi\|^3\right)$ describes terms of order 3 or higher.

5.4 Welfare

When analyzing the welfare in the model, we focus on the limiting cashless economy. The social welfare function is then

$$
\sum_{t=0}^{\infty}\beta^t SW_t, \tag{233}
$$

where

$$
SW_t=u\left(C_t,Q_t\right)-\int_0^1 v\left(L_t\left(f\right),Z_t\right)df. \tag{234}
$$

Taking a second-order approximation of $u\left(C_t,Q_t\right)$ gives

$$
\begin{aligned}
u\left(C_t,Q_t\right) &= \bar{u}+\bar{u}_C\bar{C}\left(\hat{C}_t+\frac{1}{2}\left(\hat{C}_t\right)^2\right)+\bar{u}_Q\bar{Q}\left(\hat{Q}_t+\frac{1}{2}\left(\hat{Q}_t\right)^2\right)+\frac{1}{2}\bar{u}_{CC}\bar{C}^2\left(\hat{C}_t\right)^2 \\
&+\bar{u}_{CQ}\bar{C}\bar{Q}\hat{C}_t\hat{Q}_t+\frac{1}{2}\bar{u}_{QQ}\bar{Q}^2\left(\hat{Q}_t\right)^2+o\left(\|\xi\|^3\right).
\end{aligned}
\tag{235}
$$

Let us take a second order approximation of $v\left(L_t\left(f\right),Z_t\right)$ using the standard variance decomposition $E_f\left(\hat{L}_t\right)^2=var_f\hat{L}_t+\left(E_f\hat{L}_t\right)^2$. Also, since $\hat{Z}_t$ is an aggregate variable we have $E_f\hat{Z}_t=\hat{Z}_t$. Using

(199) to eliminate $E_f \hat{L}_t^d(f)$, $E_f \hat{L}_t(f)$, $\left(E_f \hat{L}_t^d(f)\right)^2$ and $\left(E_f \hat{L}_t\right)^2$ gives

$$\begin{aligned} E_f v\left(L_t(f), Z_t\right) &= \bar{v}_L \bar{L}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t^d - \hat{A}_t\right) - \frac{1}{2}\frac{\sigma-1}{\sigma} var_f \hat{y}_t(f) + \frac{1}{2} var_f \hat{L}_t(f)\right) \\ &+ \bar{v}_L \bar{L}\frac{1}{2}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t^d - \hat{A}_t\right)\right)^2 + \bar{v}_Z \bar{Z}\left(\hat{Z}_t + \frac{1}{2}\left(\hat{Z}_t\right)^2\right) + \frac{1}{2}\bar{v}_{LL}\bar{L}^2\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)^2 \\ &+ \bar{v}_{LZ}\bar{L}\bar{Z}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)\hat{Z}_t + \frac{1}{2}\bar{v}_{ZZ}\bar{Z}^2\left(\hat{Z}_t\right)^2 + tip + o\left(\|\xi\|^3\right), \end{aligned} \tag{236}$$

where tip denotes terms that are independent of policy.

Since $\hat{Z}_t$ is and aggregate (and thus common) disturbance we have $E_f \hat{Z}_t = \hat{Z}_t$.[12]

Combining the second order approximations of $u(C_t, Q_t)$ and $E_f v\left(L_t(f), Z_t\right)$ from expressions (235) and (236), gives welfare as[13]

$$\begin{aligned} SW_t &= \bar{u}_C \bar{C}\left(\hat{C}_t + \frac{1}{2}\left(\hat{C}_t\right)^2\right) + \frac{1}{2}\bar{u}_{CC}\bar{C}^2\left(\hat{C}_t\right)^2 + \bar{u}_{CQ}\bar{C}\bar{Q}\hat{C}_t\hat{Q}_t \\ &-\bar{v}_L\bar{L}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t^d - \hat{A}_t\right) - \frac{1}{2}\frac{\sigma-1}{\sigma} var_f \hat{Y}_t(f) + \frac{1}{2} var_f \hat{L}_t(f) + \frac{1}{2}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t^d - \hat{A}_t\right)\right)^2\right) \\ &-\frac{1}{2}\bar{v}_{LL}\bar{L}^2\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)^2 - \bar{v}_{LZ}\bar{L}\bar{Z}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\right)\hat{Z}_t + tip + o\left(\|\xi\|^3\right) \end{aligned} \tag{237}$$

We are interested in computing the difference between sticky and flexible-price welfare. Consider welfare when prices are flexible. Note that there is no variance in the price and wage distribution across firms, since all prices and wages are adjusted in every period. Let us analyze the difference $SW_t - SW_t^*$, i.e. the welfare difference, using that $\hat{C}_t = \hat{Y}_t^d = \hat{Y}_t - G_t$ and $\hat{C}_t^* = \hat{Y}_t^{d*} = \hat{Y}_t^* - G_t$ to eliminate $\hat{C}_t$ and $\hat{C}_t^*$, and that flexible-price welfare is similar to SW_t, except that all variables are evaluated at flexible prices implying that the price and wage distributions across firms are degenerate (i.e. zero variance). Also, using that $\hat{Y}_t^d = \hat{Y}_t - G_t$ and $\hat{Y}_t^d - \hat{Y}_t^{d*} = \hat{Y}_t - \hat{Y}_t^*$ and that $\bar{u}_C(1-\gamma)\bar{C} = \bar{v}_L\bar{L}$ gives

$$\begin{aligned} SW_t - SW_t^* &= \left(-\bar{u}_{CC}\bar{C}^2 G_t + \bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t + \frac{\bar{v}_L\bar{L} + \bar{v}_{LL}\bar{L}^2}{(1-\gamma)^2}\hat{A}_t - \frac{\bar{v}_{LZ}\bar{L}\bar{Z}}{1-\gamma}\hat{Z}_t\right)\left(\hat{Y}_t - \hat{Y}_t^*\right) \\ &+\frac{1}{2}\left(\bar{u}_C\bar{C} - \bar{v}_L\bar{L}\left(\frac{1}{1-\gamma}\right)^2 + \bar{u}_{CC}\bar{C}^2 - \left(\frac{1}{1-\gamma}\right)^2\bar{v}_{LL}\bar{L}^2\right)\left(\left(\hat{Y}_t\right)^2 - \left(\hat{Y}_t^*\right)^2\right) \\ &-\frac{\bar{v}_L\bar{L}}{2}\left(-\frac{\sigma-1}{\sigma} var_f \hat{Y}_t(f) + var_f \hat{L}_t(f)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{238}$$

[12] Note that the terms $\left(var_f \hat{y}_t\right)^2$, $var_f \hat{y}_t var_f \hat{L}_t(f)$, $\left(var_f \hat{L}_t(f)\right)^2$, $\left(\hat{Y}_t - \hat{A}_t\right)\left(var_f \hat{y}_t\right)$, $\left(\hat{Y}_t^d - \hat{A}_t\right)\left(var_f \hat{y}_t\right)$, $\left(\hat{Y}_t^d - \hat{a}_t\right)\left(var_f \hat{L}_t(f)\right)$ and $\left(\hat{Y}_t - \hat{A}_t\right)\left(var_f \hat{L}_t(f)\right)$ appearing in the $\left(E_f \hat{L}_t^d\right)^2$ and $\left(E_f \hat{L}_t\right)^2$ terms vanish since they are of order three or higher.

[13] The terms involving only the disturbances are independent of policy.

Let us eliminate the shock terms by using that flexible-price output $\tilde{Y}_t^*$ is a function of the disturbances in the model. We define

$$\Lambda^* = \bar{u}_{CC}\bar{C}^* - \frac{(1-\gamma)\bar{Y}^*}{\overline{MPL}^2}\bar{v}_{LL}\frac{1}{1-\gamma} - \frac{1}{\overline{MPL}}\bar{v}_L\frac{\gamma}{1-\gamma} = \bar{u}_C\left(-\rho_C + \rho_L\frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma}\right), \quad (239)$$

where we use the definitions of ρ_C and ρ_L and that $\bar{u}_C\overline{MPL} = \bar{v}_L$. Recall that, using that $\widehat{mpl}_t^* = \frac{1}{1-\gamma}\hat{A}_t - \frac{\gamma}{1-\gamma}\hat{Y}_t^*$ and that $\overline{MPL} = (1-\gamma)\frac{\bar{Y}}{\bar{L}}$ we have, from expression (63)

$$\Lambda^*\bar{C}\hat{Y}_t^* = -\bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t + \bar{u}_{CC}\bar{C}^2G_t + \frac{\bar{Z}\bar{L}}{1-\gamma}\bar{v}_{LZ}\hat{Z}_t - \frac{\bar{L}}{1-\gamma}\left(\bar{v}_L + \bar{v}_{LL}\bar{L}\right)\frac{1}{1-\gamma}\hat{A}_t. \quad (240)$$

Note that, since $\bar{u}_C\overline{MPL} = \bar{v}_L$, we have $\bar{u}_C\bar{C} = \bar{v}_L\bar{L}\frac{1}{1-\gamma}$ and hence $\bar{u}_C\bar{C} - \bar{v}_L\bar{L}\left(\frac{1}{1-\gamma}\right)^2 = -\gamma\bar{v}_L\bar{L}\left(\frac{1}{1-\gamma}\right)^2$.

Using the expression above for $\hat{Y}_t^*$ in expression (238) gives

$$\begin{aligned} SW_t - SW_t^* &= -\Lambda^*\bar{C}\hat{Y}_t^*\left(\hat{Y}_t - \hat{Y}_t^*\right) + \frac{\Lambda^*\bar{C}}{2}\left(\left(\hat{Y}_t\right)^2 - \left(\hat{Y}_t^*\right)^2\right) \quad (241)\\ &\quad -\frac{\bar{v}_L\bar{L}}{2}\left(-\frac{\sigma-1}{\sigma}var_f\hat{Y}_t(f) + var_f\hat{L}_t(f)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned}$$

Note that the first row on the right hand side can be rewritten as $\frac{\Lambda\bar{C}}{2}\left(\hat{Y}_t - \hat{Y}_t^*\right)^2$. Using (241), and (210) and (211), the total welfare difference is

$$\begin{aligned} E_0\sum_{t=0}^{\infty}\beta^t\left(SW_t - SW_t^*\right) &= E_0\frac{\Lambda^*\bar{C}}{2}\sum_{t=0}^{\infty}\beta^t\left(\hat{Y}_t - \hat{Y}_t^*\right)^2 - E_0\frac{\bar{v}_L\bar{L}}{2}\sum_{t=0}^{\infty}\beta^t\sigma var_f\hat{P}_t(f) \quad (242)\\ &\quad -E_0\frac{\bar{v}_L\bar{L}}{2}\sum_{t=0}^{\infty}\beta^t\left(\gamma^2 + 2\sigma\gamma(1-\gamma)\right)var_f\hat{W}_t(f) + tip + o\left(\|\xi\|^3\right). \end{aligned}$$

Repeatedly substituting (232) into itself (forwardly), using (214), starting at period 0 gives

$$var_f\left(\log W_t(f)\right) = (\alpha_w)^{t+1}var_f\left(\log W_{-1}(f)\right) + \frac{\alpha_w}{1-\alpha_w}\sum_{s=0}^{t}\alpha_w^{t-s}\left(\hat{\pi}_s^w\right)^2 + o\left(\|\xi\|^3\right). \quad (243)$$

Multiplying by β^t on both sides, using that $var_f\left(\log W_{-1}(f)\right)$ is independent of policy and summing from period 0 to infinity gives[14]

$$E_0\sum_{t=0}^{\infty}\beta^t var_f\left(\log W_t(f)\right) = \frac{\alpha_w}{1-\alpha_w}\frac{1}{1-\alpha_w\beta}E_0\sum_{t=0}^{\infty}\beta^t\left(\hat{\pi}_t^w\right)^2 + tip + o\left(\|\xi\|^3\right). \quad (244)$$

[14] We use the following rearrangement of the double sum

$$\sum_{t=0}^{\infty}\sum_{s=0}^{t}(\alpha_w\beta)^t(\alpha_w)^{-s}\left(\hat{\pi}_s^w\right)^2 = \sum_{s=0}^{\infty}\sum_{t=s}^{\infty}(\alpha_w\beta)^t(\alpha_w)^{-s}\left(\hat{\pi}_s^w\right)^2.$$

Now consider price variability again. Expression (230) can be rewritten as

$$
\begin{aligned}
var_f\left(\log P_t(f)\right) &= \alpha_w \alpha var_f\left(\log P_{t-1}(f)\right) + (1-\alpha)\alpha_w (1-\gamma)^2 var_f \log W_{t-1}(f) \\
&\quad + \frac{\alpha_w \alpha}{(1-\alpha)\alpha_w + (1-\alpha_w)} (\hat{\pi}_t)^2 + \frac{(1-\alpha)\alpha_w (1-\gamma)^2}{(1-\alpha_w)\left((1-\alpha)\alpha_w + (1-\alpha_w)\right)} (\hat{\pi}_t^w)^2 .
\end{aligned} \tag{245}
$$

Repeatedly substituting (245) into itself (forwardly), starting at period 0 and taking expectations at period 0 gives

$$
\begin{aligned}
E_0 var_f\left(\log P_t(f)\right) &= E_0 \sum_{s=0}^{t-1} (\alpha_w \alpha)^{t-1-s} \left((1-\alpha)\alpha_w\right)(1-\gamma)^2 var_f\left(\log W_s(f)\right) \\
&\quad + E_0 \sum_{s=0}^{t} (\alpha_w \alpha)^{t-s} (1-\gamma)^2 \frac{(1-\alpha)\alpha_w}{(1-\alpha_w)\left((1-\alpha)\alpha_w + (1-\alpha_w)\right)} (\hat{\pi}_s^w)^2 \qquad (246) \\
&\quad + E_0 \sum_{s=0}^{t} (\alpha_w \alpha)^{t-s} \frac{\alpha_w \alpha}{(1-\alpha)\alpha_w + (1-\alpha_w)} (\hat{\pi}_s)^2 + tip + o\left(\|\xi\|^3\right).
\end{aligned}
$$

Multiplying by β^t on both sides and summing from period 0 to infinity gives[15]

$$
\begin{aligned}
E_0 \sum_{t=0}^{\infty} \beta^t var_f\left(\log P_t(f)\right) &= \beta \frac{(1-\alpha)\alpha_w (1-\gamma)^2}{1-\beta\alpha_w\alpha} E_0 \sum_{t=0}^{\infty} \beta^t var_f\left(\log W_t(f)\right) \\
&\quad + \frac{(1-\gamma)^2 (1-\alpha)\alpha_w}{(1-\beta\alpha_w\alpha)(1-\alpha_w)\left((1-\alpha)\alpha_w + (1-\alpha_w)\right)} E_0 \sum_{t=0}^{\infty} \beta^t (\hat{\pi}_t^w)^2 \\
&\quad + \frac{\alpha_w \alpha}{(1-\beta\alpha_w\alpha)\left((1-\alpha)\alpha_w + (1-\alpha_w)\right)} E_0 \sum_{t=0}^{\infty} \beta^t (\hat{\pi}_t)^2 + tip + o\left(\|\xi\|^3\right).
\end{aligned} \tag{247}
$$

Now we are able to state welfare in terms of squared inflation, wage inflation and output gap. From (242), (244) and (247) we get

$$
E_0 \sum_{t=0}^{\infty} \beta^t \left(SW_t - SW_t^*\right) = E_0 \sum_{t=0}^{\infty} \beta^t \mathcal{L}_t + tip + o\left(\|\xi\|^3\right), \tag{248}
$$

where

$$
\mathcal{L}_t = \theta_x (\hat{x}_t)^2 + \theta_\pi (\hat{\pi}_t)^2 + \theta_{\pi^\omega} (\hat{\pi}_t^\omega)^2, \tag{249}
$$

[15] We rewrite the double sum

$$
\begin{aligned}
\sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{t} (\alpha_w\alpha)^{t-s} &= \sum_{s=0}^{\infty}\sum_{t=s}^{\infty} \beta^t (\alpha_w\alpha)^{t-s} = \sum_{s=0}^{\infty} (\alpha_w\alpha)^{-s} \sum_{t=s}^{\infty} \beta^t (\alpha_w\alpha)^t = \sum_{s=0}^{\infty} \frac{\beta^s}{1-\beta\alpha_w\alpha}, \\
\sum_{t=0}^{\infty} \beta^t \sum_{s=0}^{t-1} (\alpha_w\alpha)^{t-1-s} &= \sum_{s=0}^{\infty}\sum_{t=s+1}^{\infty} \beta^t (\alpha_w\alpha)^{t-1-s} = \sum_{s=0}^{\infty}\sum_{r=s}^{\infty} \beta^{r+1} (\alpha_w\alpha)^{r-s} = \beta \sum_{s=0}^{\infty}\sum_{r=s}^{\infty} \beta^r (\alpha_w\alpha)^{r-s}.
\end{aligned}
$$

and, using expression (239)

$$\begin{aligned}\theta_x &= \frac{\Lambda^* \bar{C}}{2} = \frac{\bar{C}}{2}\bar{u}_C \left(-\rho_C + \rho_L \frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma}\right), \\ \theta_\pi &= -\frac{\bar{v}_L \bar{L}}{2}\sigma \frac{\alpha_w \alpha}{(1-\beta\alpha_w\alpha)\left((1-\alpha)\alpha_w + (1-\alpha_w)\right)}, \\ \theta_{\pi^\omega} &= -\frac{\bar{v}_L \bar{L}}{2}\frac{\alpha_w}{1-\alpha_w}\left(\left(\sigma\beta\frac{(1-\alpha)\alpha_w(1-\gamma)^2}{1-\beta\alpha_w\alpha} + \gamma^2 + 2\sigma\gamma(1-\gamma)\right)\frac{1}{1-\alpha_w\beta}\right. \\ &\quad \left. +\sigma\frac{(1-\gamma)^2(1-\alpha)}{(1-\beta\alpha_w\alpha)\left((1-\alpha)\alpha_w + (1-\alpha_w)\right)}\right).\end{aligned} \tag{250}$$

Using expressions (120) and (190) we get

$$\begin{aligned}\theta_x &= \frac{\Lambda^* \bar{C}}{2} = \frac{\bar{C}}{2}\bar{u}_C \left(-\rho_C + \rho_L \frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma}\right), \\ \theta_\pi &= -\frac{\bar{v}_L \bar{L}}{2}\sigma\frac{1}{\Pi}, \\ \theta_{\pi^\omega} &= -\frac{\bar{v}_L \bar{L}}{2}\left(\left(-\gamma\varepsilon_L + \sigma(1-\gamma)\right)\frac{1}{\Pi_1} - \sigma(1-\gamma)^2\frac{1}{\Pi}\right).\end{aligned} \tag{251}$$

The last coefficient can alternatively be written as

$$\theta_{\pi^\omega} = -\frac{\bar{v}_L \bar{L}}{2}\left(\gamma\left(-\varepsilon_L + \sigma(1-\gamma)\right)\frac{1}{\Pi_1} + \sigma(1-\gamma)^2\left(\frac{1}{\Pi_1} - \frac{1}{\Pi}\right)\right). \tag{252}$$

Note that $\theta_x < 0$, $\theta_\pi < 0$ and $\theta_{\pi^\omega} < 0$.

6 Optimal Discretionary Policy

To find the optimal rule under discretion, the central bank maximizes welfare in (248) subject to the constraints derived from market behavior (121) and (191), together with the real wage flow equation (194). Thus, the central bank solves the following problem

$$V\left(\hat{w}_{t-1}, \hat{w}_t^*\right) = \max_{\{\hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^\omega, \hat{w}_t\}} \theta_x \left(\hat{x}_t\right)^2 + \theta_\pi \left(\hat{\pi}_t\right)^2 + \theta_{\pi^\omega}\left(\hat{\pi}_t^\omega\right)^2 + \beta E_t V\left(\hat{w}_t, \hat{w}_{t+1}^*\right) + tip + o\left(\|\xi\|^3\right), \tag{253}$$

subject to

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + (1-\gamma)\left(\hat{\pi}_t^\omega - \beta E_t \hat{\pi}_{t+1}^\omega\right) + \Pi\left(\hat{w}_t - \hat{w}_t^*\right) + \frac{\gamma}{1-\gamma}\Pi\hat{x}_t, \tag{254}$$

$$\hat{w}_t = \hat{w}_{t-1} + \hat{\pi}_t^\omega - \hat{\pi}_t, \tag{255}$$

$$\hat{\pi}_t^\omega = \beta E_t \hat{\pi}_{t+1}^\omega - \Omega_x \hat{x}_t - \Omega_w\left(\hat{w}_t - \hat{w}_t^*\right), \tag{256}$$

where

$$\Omega_x = \Pi_1 \frac{\Phi_x}{\Phi_d}, \qquad (257)$$
$$\Omega_w = \Pi_1 \frac{\Phi_w}{\Phi_d}.$$

The first-order conditions are

$$\begin{aligned}
0 &= 2\theta_x \hat{x}_t - \lambda_t^{\pi} \frac{\gamma}{1-\gamma}\Pi + \lambda_t^{\pi^\omega}\Omega_x, \\
0 &= 2\theta_\pi \hat{\pi}_t + \lambda_t^{\pi} + \lambda_t^{w}, \qquad (258) \\
0 &= 2\theta_{\pi^\omega} \hat{\pi}_t^\omega - \lambda_t^{\pi}(1-\gamma) - \lambda_t^{w} + \lambda_t^{\pi^\omega}, \\
0 &= \beta E_t V_1\left(\hat{w}_t, \hat{w}_{t+1}^*\right) - \lambda_t^{\pi}\left(\beta E_t \frac{\partial \hat{\pi}_{t+1}}{\partial \hat{w}_t} - (1-\gamma)\beta E_t \frac{\partial \hat{\pi}_{t+1}^\omega}{\partial \hat{w}_t} + \Pi\right) + \lambda_t^{w} - \lambda_t^{\pi^\omega}\left(\beta E_t \frac{\partial \hat{\pi}_{t+1}^\omega}{\partial \hat{w}_t} - \Omega_w\right).
\end{aligned}$$

where λ_t^{π}, λ_t^{w} and $\lambda_t^{\pi^\omega}$ denotes the Lagrange multipliers from the Lagrangian. We restrict attention to Markov perfect equilibria, i.e., we do not consider any equilibria with reputational effects. However, we need to take into account that real wages is an endogenous state variable. In any stationary equilibrium, therefore, expected inflation and expected wage inflation will depend on lagged real wages. What the policy maker takes as given, accordingly, is not the level of expected inflation and wage inflation, but rather how private sector expectations of inflation and wage inflation tomorrow respond to movements in real wages today.

To solve the model we first use the three first of the first-order conditions to solve for the Lagrange multipliers as functions of the control variables. We then use the solutions to substitute into the last first-order condition, so that this first-order condition is expressed in control and state variables only. Using the first three first-order conditions gives

$$\begin{pmatrix} -\frac{\gamma}{1-\gamma}\Pi & 0 & \Omega_x \\ 1 & 1 & 0 \\ -(1-\gamma) & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} \lambda_t^{\pi} \\ \lambda_t^{w} \\ \lambda_t^{\pi^\omega} \end{pmatrix} = -2 \begin{pmatrix} \theta_x \hat{x}_t \\ \theta_\pi \hat{\pi}_t \\ \theta_{\pi^\omega} \hat{\pi}_t^\omega \end{pmatrix}. \qquad (259)$$

The solution for the Lagrange multipliers is

$$\begin{pmatrix} \lambda_t^{\pi} \\ \lambda_t^{w} \\ \lambda_t^{\pi^\omega} \end{pmatrix} = -2 \begin{pmatrix} -\frac{\gamma}{1-\gamma}\Pi & 0 & \Omega_x \\ 1 & 1 & 0 \\ -(1-\gamma) & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \theta_x \hat{x}_t \\ \theta_\pi \hat{\pi}_t \\ \theta_{\pi^\omega} \hat{\pi}_t^\omega \end{pmatrix}. \qquad (260)$$

Thus,we have now expressed the Lagrange multipliers as functions of $\hat{x}_t$, $\hat{\pi}_t$ and $\hat{\pi}_t^\omega$, and we can thus eliminate these from the last first-order condition, which is

$$\begin{aligned} 0 &= \beta E_t V_1 \left(\hat{w}_t, \hat{w}^*_{t+1}\right) - \lambda^{\pi}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \left(\beta E_t \frac{\partial \hat{\pi}_{t+1}}{\partial \hat{w}_t} - (1-\gamma)\, \beta E_t \frac{\partial \hat{\pi}^{\omega}_{t+1}}{\partial \hat{w}_t} + \Pi\right) + \lambda^{w}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \\ &\quad -\lambda^{\pi^{\omega}}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \left(\beta E_t \frac{\partial \hat{\pi}^{\omega}_{t+1}}{\partial \hat{w}_t} - \Omega_w\right). \end{aligned} \tag{261}$$

We can then solve the model by adding the constraints to the above equation and we thus have

$$\begin{aligned} 0 &= \beta E_t V_1 \left(\hat{w}_t, \hat{w}^*_{t+1}\right) - \lambda^{\pi}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \left(\beta E_t \frac{\partial \hat{\pi}_{t+1}}{\partial \hat{w}_t} - (1-\gamma)\, \beta E_t \frac{\partial \hat{\pi}^{\omega}_{t+1}}{\partial \hat{w}_t} + \Pi\right) \\ &\quad +\lambda^{w}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) - \lambda^{\pi^{\omega}}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \left(\beta E_t \frac{\partial \hat{\pi}^{\omega}_{t+1}}{\partial \hat{w}_t} - \Omega_w\right), \\ \hat{\pi}_t &= \beta E_t \hat{\pi}_{t+1} + (1-\gamma) \left(\hat{\pi}^{\omega}_t - \beta E_t \hat{\pi}^{\omega}_{t+1}\right) + \Pi \left(\hat{w}_t - \hat{w}^*_t\right) + \frac{\gamma}{1-\gamma} \Pi \hat{x}_t, \\ \hat{w}_t &= \hat{w}_{t-1} + \hat{\pi}^{\omega}_t - \hat{\pi}_t, \\ \hat{\pi}^{\omega}_t &= \beta E_t \hat{\pi}^{\omega}_{t+1} - \Omega_x \hat{x}_t - \Omega_w \left(\hat{w}_t - \hat{w}^*_t\right). \end{aligned} \tag{262}$$

This system can then be solved for optimal paths of the control variables $(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t)$ by the numerical method described in Carlsson and Westermark (2006).

Note that, using the envelope theorem, the first expression in (262) can be rewritten as

$$\begin{aligned} 0 &= \beta \left(-E_t \lambda^{w}_{t+1} \left(\hat{x}_{t+1}, \hat{\pi}_{t+1}, \hat{\pi}^{\omega}_{t+1}\right)\right) - \lambda^{\pi}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \left(\beta E_t \frac{\partial \hat{\pi}_{t+1}}{\partial \hat{w}_t} - (1-\gamma)\, \beta E_t \frac{\partial \hat{\pi}^{\omega}_{t+1}}{\partial \hat{w}_t} + \Pi\right) \\ &\quad +\lambda^{w}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) - \lambda^{\pi^{\omega}}_t \left(\hat{x}_t, \hat{\pi}_t, \hat{\pi}^{\omega}_t\right) \left(\beta E_t \frac{\partial \hat{\pi}^{\omega}_{t+1}}{\partial \hat{w}_t} - \Omega_w\right). \end{aligned} \tag{263}$$

7 The Erceg, Henderson, and Levin (2000) model

Since the sticky price equilibrium is derived as in Erceg, Henderson, and Levin (2000), we do not reproduce the derivations here. Condition (255) is identical in the two models. The conditions corresponding to (254) and (256) are

$$
\begin{aligned}
\hat{\pi}_t &= \beta E_t \hat{\pi}_{t+1} + \Pi \left(\hat{w}_t - \hat{w}_t^* \right) + \frac{\gamma}{1-\gamma} \Pi \hat{x}_t, \\
\hat{\pi}_t^{\omega} &= \beta E_t \hat{\pi}_{t+1}^{\omega} - \Omega_w^E \left(\hat{w}_t - \hat{w}_t^* \right) + \Omega_x^E \hat{x}_t
\end{aligned} \tag{264}
$$

where, Π and Π_1 are defined as in (120) and (190), respectively, and

$$
\begin{aligned}
\Omega_w^E &= \frac{\Pi_1}{1-\rho_N \sigma_w}, \\
\Omega_x^E &= \Omega_w^E \left(\rho_C - \rho_N \frac{1}{1-\gamma} \right).
\end{aligned} \tag{265}
$$

7.1 Welfare

When computing welfare in this model, a second-order approximation in logs is used, resulting in that we can relate welfare to the variance in relative prices and wages. Also, the output gap matters because it distorts the economywide relationship between consumption and leisure. Before analyzing welfare, we first compute second-order approximations of L_t and Y_t, the relationship between real variation and price variation and finally persistence in price variability.

7.2 Quadratic approximation of L_t and Y_t

We first proceed by looking at a quadratic approximation of L_t and Y_t. Aggregate demand of labor by firms is, where the integral is taken over firms

$$
L_t = \int_0^1 L(f)\, df. \tag{266}
$$

Then a quadratic approximation is

$$
\hat{L}_t = E_f \hat{L}_t(f) + \frac{1}{2} var_f \hat{L}_t(f). \tag{267}
$$

Using the definition of the composite good in (1), we can similarly derive

$$
E_f \hat{Y}_t(f) = \hat{Y}_t - \frac{1}{2} \frac{\sigma - 1}{\sigma} var_f \hat{Y}_t(f). \tag{268}
$$

Now, let us express (267) in terms of aggregate variables and variances. Composite labor in Erceg, Henderson, and Levin (2000) is given by

$$L_t = \left(\int_0^1 N_t(j)^{\frac{\sigma_w - 1}{\sigma_w}} dj \right)^{\frac{\sigma_w}{\sigma_w - 1}}. \tag{269}$$

By a similar argument to (268), we get

$$E_j \hat{N}_t(j) = \hat{L}_t - \frac{1}{2} \frac{\sigma_w - 1}{\sigma_w} var_j \hat{N}_t(j). \tag{270}$$

As in our model, we can write

$$E_f \hat{L}_t(f) = E_f \hat{Y}_t(f) - \frac{1}{1-\gamma} \hat{A}_t + \frac{\gamma}{1-\gamma} \hat{Y}_t \tag{271}$$

and, noting that capital labor ratios are the same for all firms, we have

$$var_f \hat{Y}_t(f) = var_f \hat{L}_t(f). \tag{272}$$

since there is no local variation in $\hat{A}_t$. Using (267), (268) and (272) gives

$$\hat{L}_t = E_f \hat{Y}_t(f) - \frac{1}{1-\gamma} \hat{A}_t + \frac{\gamma}{1-\gamma} \hat{Y}_t + \frac{1}{2} var_f \hat{L}_t(f) = \frac{1}{1-\gamma} \left(\hat{Y}_t - \hat{A}_t \right) + \frac{1}{2} \frac{1}{\sigma} var_f \hat{Y}_t(f). \tag{273}$$

Then, using (273), (270) can be rewritten as

$$E_j \hat{N}_t(j) = \frac{1}{1-\gamma} \left(\hat{Y}_t - \hat{A}_t \right) + \frac{1}{2} \frac{1}{\sigma} var_f \hat{Y}_t(f) - \frac{1}{2} \frac{\sigma_w - 1}{\sigma_w} var_j \hat{N}_t(j) + o\left(\|\xi\|^3 \right). \tag{274}$$

7.3 Relationship between real and price variability

Using a quadratic approximation of (3)

$$var_f \hat{Y}_t(f) = \sigma^2 var_f \hat{P}_t(f), \tag{275}$$

and similarly for labor demand, derived from (269)

$$var_j \hat{N}_t(j) = \sigma_w^2 var_j \hat{w}_t(j). \tag{276}$$

7.4 Variance Persistence

Since prices and wages are not fully flexible, the variance of the price and wage distribution across firms are persistent. We want to find the variance of the distributions today as function of previous variances and inflation. To do this, let us express $var_f(\log P_t(f))$ and $var_j(\log W_t(j))$ in terms of squared inflation and wage inflation. Combining this with (276) and (275) we get a relationship between real variability and inflation, which enables us to write welfare in terms of inflation and wage inflation. Let $\bar{P}_t = E_f \log P_t(f)$. We have, using expression (214)

$$var_f(\log P_t(f)) = E_f\left(\log P_t(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2, \tag{277}$$

We can write the variance in (277) as

$$\begin{aligned} var_f(\log P_t(f)) &= \alpha_w\alpha E_f\left(\log\bar{\pi}P_{t-1}(f) - \log\bar{\pi} - \bar{P}_{t-1}\right)^2 - \left(\Delta\bar{P}_t\right)^2 \\ &+ (1-\alpha_w\alpha)\left(\log P_t^o - \log\bar{\pi} - \bar{P}_{t-1}\right)^2. \end{aligned} \tag{278}$$

We now rewrite expression (278) in terms of lagged variance in prices and inflation. To do this, we need to rewrite the second and third term in expression (278) in terms of inflation. First, note that we have

$$\Delta\bar{P}_t = (1-\alpha_w\alpha)\left(\log P_t^o - \log\bar{\pi} - \bar{P}_{t-1}\right). \tag{279}$$

Then we have, using (214) and the expression above in (278)

$$var_f(\log P_t(f)) = \alpha_w\alpha var_f(\log P_{t-1}(f)) + \frac{\alpha_w\alpha}{1-\alpha_w\alpha}(\hat{\pi}_t)^2 + o\left(\|\xi\|^3\right). \tag{280}$$

For wages, we can write, using a similar method as when deriving (280)

$$var_j(\log W_t(j)) = \alpha_w var_j(\log W_{t-1}(j)) + \frac{\alpha_w}{1-\alpha_w}(\hat{\pi}_t^w)^2 + o\left(\|\xi\|^3\right), \tag{281}$$

where $o\left(\|\xi\|^3\right)$ describes terms of order 3 or higher.

7.5 Welfare

When analyzing the welfare in the model, we focus on the limiting cashless economy. The social welfare function is then

$$\sum_{t=0}^{\infty}\beta^t SW_t, \tag{282}$$

with SW_t defined as

$$SW_t = u\left(C_t, Q_t\right) - \int_0^1 v\left(N_t\left(j\right), Z_t\right) dj. \tag{283}$$

Taking a second-order approximation of $u\left(C_t, Q_t\right)$ is identical to our model; see expression (235).

Loglinearizing the second term in (283) gives, using the standard variance decomposition $E_j\left(\hat{N}_t\right)^2 = var_j \hat{N}_t\left(j\right) + \left(E_j \hat{N}_t\right)^2$ and expression (274) for $\hat{N}_t$ and $\hat{N}_t^d$, respectively. Since G_t is aggregate and we have $var_f \hat{Y}_t^d\left(f\right) = var_f \hat{Y}_t\left(f\right)$ and $var_j \hat{N}_t^d\left(j\right) = var_j \hat{N}_t\left(j\right)$. The aggregate shock does not affect in between firm variance. Since $\hat{Z}_t$ is aggregate we have $E_j \hat{Z}_t = \hat{Z}_t$ and hence[16]

$$\begin{aligned} E_j v\left(N_t\left(j\right), Z_t\right) &= \bar{v}_N \bar{N}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t^d - \hat{A}_t\right) + \frac{1}{2}\frac{1}{\sigma} var_f \hat{Y}_t\left(f\right) - \frac{1}{2}\frac{\sigma_w - 1}{\sigma_w} var_j \hat{N}_t\left(j\right)\right) \\ &+ \bar{v}_N \bar{N}\left(\frac{1}{2}\left(var_j \hat{N}_t\left(j\right) + \left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t^d - \hat{A}_t\right)^2\right)\right) + \bar{v}_Z \bar{Z}\left(\hat{Z}_t + \frac{1}{2}\left(\hat{Z}_t\right)^2\right) \\ &+ \frac{1}{2}\bar{v}_{NN}\bar{N}^2\left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t - \hat{A}_t\right)^2 + \bar{v}_{NZ}\bar{N}\bar{Z}\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\hat{Z}_t \\ &+ \frac{1}{2}\bar{v}_{ZZ}\bar{Z}^2\left(\hat{Z}_t\right)^2 + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{284}$$

Combining the log linearizations of $u\left(C_t^j, Q_t\right)$ and $\int_0^1 v\left(N_t\left(j\right), Z_t\right) dj$ from expressions (235) and (284) gives welfare as

$$\begin{aligned} SW_t &= \bar{u}_C \bar{C}\left(\hat{C}_t + \frac{1}{2}\left(\hat{C}_t\right)^2\right) + \frac{1}{2}\bar{u}_{CC}\bar{C}^2\left(\hat{C}_t\right)^2 + \bar{u}_{CQ}\bar{C}\bar{Q}\hat{C}_t\hat{Q}_t \\ &- \bar{v}_N \bar{N}\left(\frac{1}{1-\gamma}\left(\hat{Y}_t^d - \hat{A}_t\right) + \frac{1}{2}\frac{1}{\sigma} var_f \hat{Y}_t\left(f\right) + \frac{1}{2}\frac{1}{\sigma_w} var_j \hat{N}_t\left(j\right) + \frac{1}{2}\left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t^d - \hat{A}_t\right)^2\right) \\ &- \frac{1}{2}\bar{v}_{NN}\bar{N}^2\left(\frac{1}{1-\gamma}\right)^2\left(\hat{Y}_t - \hat{A}_t\right)^2 - \bar{v}_{nZ}\bar{N}\bar{Z}\frac{1}{1-\gamma}\left(\hat{Y}_t - \hat{A}_t\right)\hat{Z}_t + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{285}$$

We are interested in computing the difference between sticky and flexible price welfare. The difference $SW_t - SW_t^*$ is, using that $\bar{u}_C \bar{C}\left(1-\gamma\right) = \bar{v}_N \bar{N}$

$$\begin{aligned} SW_t - SW_t^* &= \left(-\bar{u}_{CC}\bar{C}^2 G_t + \bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t + \frac{\bar{v}_N \bar{N} + \bar{v}_{NN}\bar{N}^2}{\left(1-\gamma\right)^2}\hat{A}_t - \frac{\bar{v}_{NZ}\bar{N}\bar{Z}}{1-\gamma}\hat{Z}_t\right)\left(\hat{Y}_t - \hat{Y}_t^*\right) \\ &+ \frac{1}{2}\left(\bar{u}_C \bar{C} - \left(\frac{1}{1-\gamma}\right)^2 \bar{v}_N \bar{N} + \bar{u}_{CC}\bar{C}^2 - \bar{v}_{NN}\bar{N}^2\right)\left(\left(\hat{Y}_t\right)^2 - \left(\hat{Y}_t^*\right)^2\right) \\ &- \bar{v}_N \bar{N}\left(\frac{1}{2}\frac{1}{\sigma} var_f \hat{Y}_t\left(f\right) + \frac{1}{2}\frac{1}{\sigma_w} var_j \hat{N}_t\left(j\right)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{286}$$

[16] Note that the terms $\left(var_f \hat{Y}_t\left(f\right)\right)^2$, $var_f \hat{Y}_t\left(f\right) var_j \hat{N}_t\left(j\right)$, $\left(var_j \hat{N}_t\left(j\right)\right)^2$, $\left(\hat{Y}_t - \hat{A}_t\right)\left(var_f \hat{Y}_t\left(f\right)\right)$, $\left(\hat{Y}_t^d - \hat{A}_t\right)\left(var_f \hat{Y}_t\left(f\right)\right)$, $\left(\hat{Y}_t^d - \hat{A}_t\right)\left(var_j \hat{N}_t\left(j\right)\right)$ and $\left(\hat{Y}_t - \hat{A}_t\right)\left(var_j \hat{N}_t\left(j\right)\right)$ appearing in the $\left(E_j \hat{N}_t^d\left(j\right)\right)^2$ and $\left(E_j \hat{N}_t\left(j\right)\right)^2$ terms vanish since they are of order 3 and higher.

We can eliminate the shock terms by using that flexible price output $\tilde{Y}_t^*$ can be written as a function of shocks. As in our model, see expression (240), we can write

$$\Lambda^* \bar{C}\hat{Y}_t^* = -\bar{u}_{CQ}\bar{C}\bar{Q}\hat{Q}_t + \bar{u}_{CC}\bar{C}^2 G_t + \frac{\bar{Z}\bar{N}}{(1-\gamma)}\bar{v}_{NZ}\hat{Z}_t - \frac{\bar{N}}{1-\gamma}\bar{v}\left(_N + \bar{v}_{NN}\bar{N}\right)\frac{1}{1-\gamma}\hat{A}_t \tag{287}$$

where

$$\Lambda^* = \bar{u}_{CC}\bar{C}^* - \bar{v}_{NN}\frac{\bar{N}}{\overline{MPL}}\frac{1}{1-\gamma} - \frac{1}{\overline{MPL}}\bar{v}_N\frac{\gamma}{1-\gamma}. \tag{288}$$

Using the expression above for $\Lambda^*\bar{C}\hat{Y}_t^*$ in expression (286) for $SW_t - SW_t^*$ gives

$$\begin{aligned} SW_t - SW_t^* \quad &= \quad \frac{\Lambda^*\bar{C}}{2}\left(\hat{Y}_t - \hat{Y}_t^*\right)^2 \\ &\quad -\bar{v}_N\bar{N}\frac{1}{2}\left(\frac{1}{\sigma}var_f\hat{Y}_t(f) + \frac{1}{\sigma_w}var_j\hat{N}_t(j)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{289}$$

Using (289), and (276) and (275), the total welfare difference is

$$\begin{aligned} \sum_{t=0}^{\infty}\beta^t\left(SW_t - SW_t^*\right) \quad &= \quad \frac{\Lambda\bar{C}}{2}\sum_{t=0}^{\infty}\beta^t\left(\hat{Y}_t - \hat{Y}_t^*\right)^2 \\ &\quad -\bar{v}_n\bar{N}\frac{1}{2}\sum_{t=0}^{\infty}\beta^t\left(\sigma var_f\hat{P}_t(f) + \sigma_w var_j\hat{W}_t(j)\right) + tip + o\left(\|\xi\|^3\right). \end{aligned} \tag{290}$$

Repeatedly substituting expression (280) into itself (forwardly), starting at 0 gives

$$var_f\left(\log P_t(f)\right) = \left(\alpha_w\alpha\right)^{t+1}var_f\left(\log P_{-1}(f)\right) + \frac{\alpha_w\alpha}{1-\alpha_w\alpha}\sum_{s=0}^{t}\left(\alpha_w\alpha\right)^{t-s}\left(\hat{\pi}_s\right)^2 + o\left(\|\xi\|^3\right). \tag{291}$$

Multiplying by β^t on both sides and summing from 0 to infinity gives

$$\sum_{t=0}^{\infty}\beta^t var_f\left(\log P_t(f)\right) = \frac{\alpha_w\alpha}{1-\alpha_w\alpha}\frac{1}{1-\alpha_w\alpha\beta}\sum_{t=0}^{\infty}(\beta)^t\left(\hat{\pi}_t\right)^2 + tip + o\left(\|\xi\|^3\right). \tag{292}$$

The same can be done for $var_j\left(\log W_t(j)\right)$. We get

$$\sum_{t=0}^{\infty}\beta^t var_f\left(\log W_t(f)\right) = \frac{\alpha_w}{1-\alpha_w}\frac{1}{1-\alpha_w\beta}\sum_{t=0}^{\infty}(\beta)^t\left(\hat{\pi}_t^{\omega}\right)^2 + tip + o\left(\|\xi\|^3\right). \tag{293}$$

Using expressions (292) and (293) in (290) gives

$$\sum_{t=0}^{\infty}\beta^t\left(SW_t - SW_t^*\right) = \sum_{t=0}^{\infty}\beta^t L_t + tip + o\left(\|\xi\|^3\right) \tag{294}$$

where

$$L_t = \theta_x (\hat{x}_t)^2 + \theta_\pi (\hat{\pi}_t)^2 + \theta_{\pi^\omega} (\hat{\pi}_t^\omega)^2 \tag{295}$$

and

$$\begin{aligned}
\theta_x &= \frac{\Lambda^* \bar{C}}{2} = \frac{\bar{u}_C \bar{C}}{2} \left(-\rho_C + \rho_N \frac{1}{1-\gamma} - \frac{\gamma}{1-\gamma} \right), \\
\theta_\pi &= -\frac{1}{2} \bar{\upsilon}_N \bar{N} \frac{1}{\Pi} \sigma, \\
\theta_\omega &= -\frac{1}{2} \bar{\upsilon}_N \bar{N} \frac{1}{\Pi_1} \sigma_w.
\end{aligned} \tag{296}$$

7.6 Optimal Policy

To find the optimal rule under discretion, the central bank solves the following problem

$$V(\hat{w}_{t-1}, \hat{w}_t^*) = \max_{\{\hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^\omega, \hat{w}_t\}} \theta_x (\hat{x}_t)^2 + \theta_\pi (\hat{\pi}_t)^2 + \theta_{\pi^\omega} (\hat{\pi}_t^\omega)^2 + \beta E_t V\left(\hat{w}_t, \hat{w}_{t+1}^*\right) + tip + o\left(\|\xi\|^3\right) \tag{297}$$

subject to

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \Pi (\hat{w}_t - \hat{w}_t^*) + \frac{\gamma}{1-\gamma} \Pi \hat{x}_t, \tag{298}$$

$$\hat{w}_t = \hat{w}_{t-1} + \hat{\pi}_t^\omega - \hat{\pi}_t, \tag{299}$$

$$\hat{\pi}_t^\omega = \beta E_t \hat{\pi}_{t+1}^\omega - \Omega_w^E (\hat{w}_t - \hat{w}_t^*) + \Omega_x^E \hat{x}_t. \tag{300}$$

As in our model, we can write the Lagrange multipliers as functions of $\hat{x}_t$, $\hat{\pi}_t$ and $\hat{\pi}_t^\omega$

$$\begin{pmatrix} \lambda_t^\pi \\ \lambda_t^w \\ \mu_t^{\pi^\omega} \end{pmatrix} = -2 \begin{pmatrix} -\frac{\gamma}{1-\gamma}\Pi & 0 & -\Omega_x^E \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \theta_x \hat{x}_t \\ \theta_\pi \hat{\pi}_t \\ \theta_{\pi^\omega} \hat{\pi}_t^\omega \end{pmatrix}. \tag{301}$$

Then we can eliminate the Lagrange multipliers from the first-order condition with respect to $\hat{w}_t$. We then get the following system of equations

$$\begin{aligned}
0 &= \beta E_t V_1\left(\hat{w}_t, \hat{w}_{t+1}^*\right) - \lambda_t^\pi (\hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^\omega) \left(\Pi + \beta \frac{\partial E_t \hat{\pi}_{t+1}}{\partial \hat{w}_t} \right) + \lambda_t^w (\hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^\omega) \\
&\quad - \lambda_t^{\pi^\omega} (\hat{x}_t, \hat{\pi}_t, \hat{\pi}_t^\omega) \left(\beta \frac{\partial E_t \hat{\pi}_{t+1}^\omega}{\partial \hat{w}_t} - \Omega_w^E \right), \\
\hat{\pi}_t &= \beta E_t \hat{\pi}_{t+1} + \frac{\gamma}{1-\gamma} \Pi \hat{x}_t + \Pi (\hat{w}_t - \hat{w}_t^*), \\
\hat{w}_t &= \hat{w}_{t-1} + \hat{\pi}_t^\omega - \hat{\pi}_t, \\
\hat{\pi}_t^\omega &= \beta E_t \hat{\pi}_{t+1}^\omega - \Omega_w^E (\hat{w}_t - \hat{w}_t^*) + \Omega_x^E \hat{x}_t.
\end{aligned} \tag{302}$$

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