Money and Modern Bank Runs

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Abstract

Following Diamond and Dybvig (1983), bank runs in the literature take the form of withdrawals of demand deposits payable in real goods, which deplete a fixed reserve of goods in the banking system. This paper examines modern bank runs, in which withdrawals typically take the form of wire transfers by large depositors. These transfers shift balances among banks, with no analog of a depletion of a scarce reserve from the banking system. I show that with demand deposits payable in money using modern payment systems, panic runs do not occur if there is efficient lending among banks. Aggregate shocks also do not cause bank runs because nominal deposits allow consumption to adjust efficiently with prices. Additionally, currency withdrawals do not cause traditional consumer runs unless all banks are subject to panics. However, if interbank lending breaks down, bank runs occur due to a coordination failure in which banks do not lend to a bank in need, and can lead to price deflation and contagion to other banks being run. Policy conclusions such as deposit insurance and suspension of convertibility that solve depositor-based runs, as in Diamond-Dybvig, are neither necessary nor sufficient to prevent interbank-based banking crises. Rather, central bank intervention as lender of last resort is necessary. The model corresponds to evidence of the banking crisis that required unprecedented Federal Reserve intervention following September 11, 2001.

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1. Introduction

The modern theory of bank runs is being applied to examine an increasing number of phenomena, ranging from emerging market financial crises[1] to contagion in the interbank lending market[2]. Starting with Diamond and Dybvig (1983), the common cause of bank runs throughout the literature is excessive withdrawals of demand deposits payable in real goods, which deplete a fixed reserve of real goods in the banking system.[3] This describes traditional depositor runs, such as those in the 19th and early 20th century, in which consumers withdraw and then store currency. Gorton (1988) shows that during banking panics in this era, the fraction of currency to deposits increased. There was a depletion of currency from the banking system, implying that currency was withdrawn from banks and hoarded outside of the banking system.[4]

However, in modern-economy bank runs, withdrawals take the form of wire transfers of money between banks. While money balances shift among banks, there is no analog of a depletion of a scarce reserve in such runs. Hence, the standard banking literature cannot explain this type of run. These runs may occur when wholesalers or large depositors do not roll over deposits such as CDs at a bank and deposit the funds elsewhere. Alternatively, they may flee from banks altogether by buying government or other financial securities. The large depositor sends money from his account at his current bank to either his new account at a different bank or to the bank account of the party who is selling securities. Payment for this transaction is made in the form of bank balances that are held at the central bank (or a clearinghouse) being transferred from one bank to another and there is no depletion of money from the banking system. Any funds paid by one bank (on behalf of itself or a depositor's account) are received by another bank and are available to be paid or lent out again the same day. The bank run occurs if banks that receive excess funds do not lend to the bank that has had large withdrawals and needs to borrow.

To study these modern bank runs, this paper focuses on deposits payable in money using electronic payment systems to highlight the role of interbank lending. I argue that the main risk of modern banking fragility lies in the interbank lending market, because if interbank lending is efficient, I show that panic bank runs do not occur with modern payment systems. Any money withdrawn from one bank is transferred to another bank, so if it is lent back to the

[1] See Allen and Gale (2000b), Chang and Velasco (1998, 2000) and Gale and Vives (2002).

[2] See Allen and Gale (2000a), Diamond and Rajan (2003a) and Freixas et al. (2000).

[3] An exception is Diamond and Rajan (2003b) discussed below.

[4] Several banking papers argue that models with demand deposit contracts that pay real goods describe bank runs during particular eras (in which banks in actuality paid demand deposit withdrawals in money). Allen and Gale (1998) cite the 19th and early 20th century bank runs in Gorton (1988). Diamond and Rajan (2003a) appeal to the Gold Standard era.

former, the bank never fails. When interbank lending breaks down, bank failures occur which induce runs. Furthermore, even if a currency run on a bank is allowed for, if depositors are willing to redeposit at another bank, then they do not prefer to demand currency withdrawals and the run does not materialize. Since efficient interbank lending may preclude depositor runs, the main concern for modern bank runs should be directed at the interbank level.

To explain in more detail, it is helpful to review the underlying cause of bank runs in the standard literature more carefully. Starting with Diamond and Dybvig (1983), modern banking theory argues that a bank provides liquidity insurance to consumers over the uncertain timing of their consumption needs. Consumers deposit their goods in a bank for demand deposit contracts that pay a fixed amount of goods whenever the consumers choose to withdraw. The bank stores some of the goods so that they are liquid to pay to early consumers who are hit by a liquidity shock and need to consume at an early date. The bank is able to provide liquidity insurance to consumers because demand deposits pay promised fixed payments to any consumer claiming to be an early type. Meanwhile, the bank has invested the remaining goods for a greater return to pay out to the remaining late consumers who consume at a later date. However, if all consumers expect a run to occur, they all withdraw early and the bank has to inefficiently liquidate invested goods, leaving all consumers worse off.

Precisely because demand deposits are fixed payments of goods, the bank cannot ration payments to consumers to avoid a potential bank run. The excess early withdrawals of goods by late consumers necessarily causes *leakage*, the depletion of goods from the bank beyond the supply held for early consumers. To pay these withdrawals, long term investments must be liquidated, which creates further irreversible leakage due to losses from inefficient liquidation. This also implies that any late consumers who wait to withdraw at the later period cannot be fully repaid, so all late consumers join in the run and it is a self-fulfilling prophesy. Allowing for interbank lending does not by itself solve the problem, as shown in several earlier works.[5] If the aggregate amount of early withdrawals is greater than the total amount of liquid goods held by all banks, there are not enough liquid goods available to lend, so there is leakage from the banking system as a whole and bank runs result. In sum, the bank's ability to provide liquidity insurance and the bank's susceptibility to bank runs are intrinsically tied together based on fixed promises of goods.

A key driver in this theory of bank runs that often goes unexamined is that demand deposits are real-goods contracts. Since the bank has to pay consumers in consumption goods, excessive early withdrawals necessarily imply that there is leakage and the bank must liquidate investments. Hence, in the standard banking literature, bank runs ultimately occur due to leakage.

[5] See Bhattacharya and Gale (1987), Bhattacharya and Fulghieri (1994) and Allen and Gale (2000a).

This paper shows how money deposits break the link between demand deposits and fixed promises of goods. If there are excess early withdrawals of money by late consumers from a bank, the money is redeposited at a second bank. If the second bank lends the funds back to the first bank within the same period, the first bank does not have to liquidate long term loans and there is no run. Without money deposits, when there are excess early withdrawals by late consumers, the first bank must pay in goods. The bank must liquidate invested goods in order to pay. Even if the late consumers redeposit the goods at the second bank, there is already an irreversible leakage of goods from the banking system due to the costly liquidation, so bank runs are unavoidable. Thus, in order to examine how efficient interbank lending prevents bank runs, money must be introduced.

Moreover, if the second bank receives money deposits and does not lend back to the first bank, the first bank defaults on paying its withdrawals and there is a run, but this occurs without leakage from the bank or the banking system. Investments are made by entrepreneurs who have to inefficiently liquidate projects when their loans are called by a failing bank. However, the bank itself holds no goods or investments which are subject to depletion or liquidation. Rather, the bank defaults due to the mismatch in its current cash flows: inflows from current loan repayments and borrowing in the interbank market are less than outflows paid for withdrawals. In the case of real-goods deposits, runs only occur due to leakage from the banking system. Hence, in order to examine how bank runs occur without leakage due to inefficient interbank lending, again money must be introduced.

The benchmark result shows that in a model with only electronic payments and no leakage due to currency withdrawals, if there is efficient lending among banks, there are no bank runs. Specifically, the benchmark model is similar to that which has become standard since Diamond and Dybvig (1983) but with the addition of nominal contracts payable in money for demand deposits, loans to entrepreneurs and loans among banks. At the beginning of a period, a consumer can withdraw funds from his demand deposit account by sending an electronic payment from his bank to a second bank. The funds are deposited in the consumer's new account to redeposit or in the entrepreneur's account to purchase goods. Within the same period, entrepreneurs can repay loans to the original bank, and the second bank can lend surplus funds to the original bank when it is in need. An electronically instantaneous method of payment, as occurs in interbank payment systems, means that consumers do not need to hold currency in advance in order to make payments as in the cash-in-advance literature.

Without loss of generality, the interbank market is modeled as the second bank, so its role is to represent interbank payment transactions and lending with the original bank. If there is a potential run on the original bank, all funds paid by the original bank are received by the second bank, which is able to lend the funds back within the same period at no risk, since the original bank has a liquidity run and no solvency risk.

The assumption of efficient interbank lending means that a bank in need of funds can borrow in full from banks with an excess of funds if the loan will be repaid. Goodfriend and King (1988) argue that sophisticated interbank markets in modern economies are very efficient at monitoring and lending to banks in need. If the interbank market lends efficiently, theoretically it can lend funds indefinitely to allow a bank to survive potential depositor runs, assuming there is no solvency concern. Thus, efficient interbank lending implies the original bank receives the loan and does not default.

However, just because a bank does not default does not mean that it provides consumption efficiently for consumers and that there are not equilibrium runs. I show that when a bank provides optimal liquidity storage and pays fixed money payments to consumers, the market provides the optimal allocation of goods among consumers. If late consumers were to withdraw early to purchase goods, the marginal late consumer prefers not to. An abundance of consumers with money looking to buy limited goods drives the price up, so the price mechanism in the goods market rations consumption. A potential consumption run is resolved by prices adjusting in the goods market. If other late consumers were to run the bank and redeposit at the second bank, the marginal late consumer would not since the bank does not default and survives to pay late withdrawals. Thus, a potential bank run unwinds as well, and there are no runs in equilibrium.

I also show that aggregate shocks to investment returns do not cause bank runs. The benchmark model is adapted to a framework with aggregate uncertainty of returns, as in Allen and Gale (1998), but with fully nominal contracts. Allen and Gale (1998) argue that bank runs are in fact efficient responses to macroeconomic fundamental shocks, rather than pure panics, and are necessary to implement optimal risk sharing. When future returns will be low, some late consumers run the bank to share in the relative abundance of goods provided in the early period. Payments on demand deposits, whether in goods or money, are stored by late consumers who run the bank. This implies that these bank runs also require leakage.

With nominal contracts and money introduced at the beginning of the timeline in my model, the real liquidity shock of low returns on investments is resolved through prices in the goods market. Lower future returns would imply higher future prices because the same amount of money is chasing fewer goods, so entrepreneurs store goods according to profit maximization and price signals in order to allow early and late consumers to share optimally. Late consumers do not need to run the bank in order to share in first period goods. The real shock to returns does not translate into a liquidity shock to the bank, in which the bank is short of funds and needs to borrow from other banks, because nominal claims give the flexibility for consumers' real consumption to adjust with prices. I show that bank runs and central bank intervention do not occur and are not necessary for banks to achieve the optimal outcome.

I extend the benchmark model to allow for transactions with currency and storage of currency. This allows for the possibility of Diamond-Dybvig type depositor runs due to leakage in which consumers withdraw and hoard currency outside of the banking system. However, any late consumers who withdraw currency would prefer to redeposit it at the second bank since the second bank has no original depositors or illiquid long term loans issued, so it is always safe from being run and failing. Since no late consumers hoard currency, they in turn do not prefer to demand currency from the original bank since they are satisfied with electronic redeposits at the second bank. Thus, the original bank can pay and borrow electronic funds from the second bank and does not fail. Hence, the potential bank run unwinds. Therefore, allowing for leakage of currency does not necessarily cause bank runs. The conditions that allow for a depositor run are that all banks hold deposits and illiquid loans from the beginning of the timeline so that they are all subject to runs. In addition, late consumers must believe all banks are being run, and late consumers must demand currency for withdrawal. In sum, if late consumers are willing to redeposit currency anywhere, then Diamond-Dybvig runs do not materialize. The point is that while depositor-based runs could occur, they are not the only type of run and not the most relevant to modern banking systems where the primary threat of withdrawals is by large institutional depositors transferring funds by wire. However, if currency runs were to occur, the central bank could resolve them by acting as lender of last resort to just a single bank. This bank would attract currency withdrawn from banks being run and the panic would end.

Finally, I examine multiple banks in the interbank lending market to show that bank runs do occur without leakage if interbank lending is inefficient. When a bank needs to borrow from multiple banks after an exogenous shock to the banking system, there is a lending coordination problem among banks, where either all or none lend to the bank in need. A breakdown in interbank lending causes the bank in need to liquidate long term loans and default on late withdrawals. All banks lose liquidity, and consumers of all banks have suboptimal consumption sharing. If the shock is large enough, the bank in need is run. Liquidation causes price deflation and can cause contagion that leads to runs at other banks. Though the manifestation of the interbank breakdown is an actual run on the banks by depositors, the ultimate cause is the interbank market lending crisis. Due to the potential contagion of runs to other banks, I also refer to this modern bank run as a banking crisis.

The coordination failure, in which banks stop lending in the interbank market because others are not lending, is analogous to Diamond-Dybvig in which late consumers run the bank because others run. However, policy conclusions such as deposit insurance and suspension of convertibility that solve depositor-based runs, as in Diamond-Dybvig, are neither necessary nor sufficient to prevent interbank market crises. Rather, central bank intervention as lender of last resort to recoordinate lending is necessary. A lender of last resort is contentious due to moral hazard and the difficulty of identifying liquidity versus idiosyncratic solvency issues, which are not examined in this paper. Nevertheless, the need for a lender of last resort as argued in this paper may be demonstrated from the banking crisis that occurred after September 11, 2001.

Evidence from McAndrews and Potter (2002) suggests that there was an extended banking crisis following September 11 which the model may help explain. Empirical evidence suggests that after bank payment systems recovered from damage, there continued to be a protracted breakdown in interbank lending that may be explained by the model as a lending coordination failure caused by the initial displacement of bank reserves. As was necessary, the Federal Reserve lent enormous sums directly to banks in need. However, evidence suggests that the Fed's ability to restore the interbank market by acting as guaranteed lender of last resort may have been hampered due to initial uncertainty and possible reluctance over borrowing through the Fed discount window. This paper highlights the importance of a credible lender of last resort, and the recent steps leading toward this with new Fed discount window policy (as of 2003) of lending to banks without scrutiny at above market rates.

The paper proceeds as follows. Section 2 further discusses how this paper relates to the banking literature and to the institutional details of modern interbank payments and lending. The benchmark model is presented in Section 3, and the primary result of a unique first best equilibrium with no bank runs is given in Section 4. Section 5 shows that bank runs do not occur in a framework of aggregate uncertainty over returns. The extension to currency is analyzed in Section 6. Banking crises due to an inefficient interbank market are developed in Section 7, and the model is applied to the banking crisis after September 11, 2001. The final section discusses policy implications and concludes.

2. Related Literature and Institutional Details

Banking models that add interbank lending or money typically ignore the point that except for leakage due to currency storage by individuals or firms, money is not withdrawn from the economy. If there is no currency hoarding, the money supply held by banks does not shrink when demand deposits are withdrawn. Rather, the money supply is constant, so for every bank deficit there exists another bank surplus which can be lent. Banking models with interbank markets without money include those by Bhattacharya and Gale (1987), Bhattacharya and Fulghieri (1994) and Allen and Gale (2000a). Deposit claims are paid in goods, which are consumed.

Recent banking papers with interbank lending, but in which money is not modeled in general equilibrium, include those by Gale and Vives (2002), Freixas et al. (2000, 2003), Freixas and Holthausen (2001), and Rochet and Vives (2002). These papers examine the role of lending money between banks, central bank lending and injections of money, and demand deposits paid in money to consumers. However, these papers do not make a true distinction between money and goods. Money paid for deposit withdrawals is either consumed or withdrawn from the economy, just as goods are consumed in models in which goods are withdrawn. Also, many papers that consider the "good" in the economy to be currency assume in reduced form that the currency is invested and returns larger amounts of currency without an explanation of how new currency is generated. These models also do not discriminate between currency exchanged as payments and the real goods presumably purchased with the currency and invested. Currency is destroyed when a bank liquidates an investment, and is withdrawn from the economy when the currency is consumed. Frequently in these papers, money is considered an endowment with no general equilibrium explanation of where it comes from or its basis for value. There exists no separate market for goods, and utility is derived from quantities of money consumed, but with no regard for the price level or the real value of money.

In a model of emerging markets, Chang and Velasco (2000) consider dollars as goods and local currency as money. They acknowledge that their assumption that local currency is only a factor in the consumer's utility constraint is objectionable. However, they do correct for the price level. As above, they assume local currency is stored outside of the banking system and when spent is withdrawn from the economy. Since emerging market countries often have less developed interbank payment and lending systems and bank liabilities in foreign currency, which are easily leaked out of the banking system and the country, my model does not strongly apply. However, Skeie (2003b) applies the model in this paper of bank deposits payable in money versus goods to study the optimal structure of bank liabilities payable in foreign versus local currency during emerging market crises.

As described in the introduction, Allen and Gale (1998 and also 2000b) argue that shocks to aggregate returns cause bank runs, but I show that with nominal contracts, prices in the goods market preclude runs. Allen and Gale (1998) introduce money and nominal contracts, but only in a partial way. They allow for nominal demand deposit contracts to show that the central bank can inject money to inflate deposit claims in order to avoid bank runs from being destructive of value when they would be otherwise inefficient. However, money is only added to the economy when a run occurs, and money is stored by the late consumers who run and is not recirculated within the interbank market. I show the reason the bank run needs to occur in the first place is that initial demand deposit claims are not nominal contracts and there is no money in the economy at the start of the timeline. Rather, payments on demand deposits are consumed and withdrawn from the economy.

Diamond and Rajan (2003b) is an important exception to the standard bank run literature.

They are the first to introduce nominal contracts and money in general equilibrium in a model of bank runs, and they have results related to this paper. My paper shows similar to Diamond and Rajan (2003b) that nominal deposits protect banks from aggregate shocks. I also show that efficient lending in the interbank market precludes depositor panic runs. This is based on a same-period payment-in-advance constraint, in which banks can pay any amount of money that they will receive back within the same period, in order to model the interbank payment system. This allows for studying purchases and redeposits made with bank deposits, in which the interbank payment system transfers funds electronically and nearly instantaneously so that funds for large purchases and loans are paid and cleared the same day. Diamond and Rajan (2003b) assume a standard cash-in-advance constraint exists, where payments must be made a period before the purchase. They assume depositors coordinate on the optimal equilibrium and do not consider panic runs or the ability of the interbank market to preclude runs.

The primary emphasis on how bank runs may occur in Diamond and Rajan (2003b) is due to currency leakage from the banking system. Runs occur due to withdrawals of currency out of banks that is required to purchase cash goods. What is new is that they provide a general equilibrium model of this currency leakage and its implications. I show how bank runs occur without currency leakage from the banking system. Runs in Diamond and Rajan (2003b) could also occur due to aggregate delays in real production when banks also have idiosyncratic delays. I show that runs may occur due to the breakdown of the interbank lending system, caused only by a idiosyncratic financial liquidity shock, with no aggregate real shock required.

Contagion in Diamond and Rajan (2003b) is also caused by withdrawal of currency out of the banking system. Bank runs propagate due to deflation in the prices of cash goods. I show how bank runs can propagate through the aggregate price level for all goods without currency leakage. A breakdown in interbank lending leads to a bank run and liquidation, causing price deflation. Under nominal contracts, deflation propagates liquidity shortages and may cause contagion of bank runs.

This paper also relates to the literature on bank reserves and the interbank payment system, such as Henckel (1999), Flannery (1996) and Hancock and Wilcox (1996). For simplicity, I assume in the benchmark model that banks hold zero net reserves. This is not far from reality. Woodford (2000) points out that the U.K., Canada, Australia, New Zealand and Sweden among other countries have no reserve requirements and so banks hold very low reserves. For example, the central bank target for voluntarily held bank reserves in Canada is typically \$200 million and in New Zealand is typically \$20 million.

A key feature of my model is that at the beginning of a period, the bank goes into deficit from making payments on behalf of depositors withdrawing, and nets to a zero balance by receiving payments or loans back from the second bank by the end of the period. This is how the interbank payment and lending system works in practice, either through a central bank clearing system or through a private clearinghouse, i.e. systems that organize the transfer of payments between banks. A private clearinghouse typically has a netting system, which means that any payments can be made electronically during a day from one bank to another, and then all payments at the end of the day are netted against each other. If any bank will receive fewer funds than it has transferred, it can borrow funds overnight from a bank with excess funds in the interbank lending market to cover its payments. A central bank clearing system typically has a gross payment system, which means that a bank's payments and receipts are instantly credited. The central bank implicitly guarantees payments by banks and banks may have negative intraday reserve balances.

In either a private or central bank clearing system, a bank may have a charge or maximum amount allowed on negative balances carried throughout the day. However, banks borrow throughout the day to maintain their reserve balances. If the interbank market is frictionless, all banks can borrow and lend to offset any payments in order to maintain zero balances at all times during the day. Woodford (2000) points out that this is achievable through a sufficiently efficient interbank market.[6]

The model demonstrates the importance of viewing money as a process of flows for tallying credits and debits rather than as a stock. Large monetary payments flow between banks even though no stock of money is held. While the net stock of money is constant, the size of payment flows is endogenous depending on the activity of banks and the economy. This corresponds to the enormous circulation of money in clearing systems in the U.S. In 1998, the average daily total payments over Fedwire (the Federal Reserve's clearing system) was \$1.2 trillion (Furfine, 1999) made on a base of bank reserve balances of only \$15 billion (McAndrews and Potter, 2002). The model also explains the large size of money flows relative to net payments between banks. In equilibrium, there are gross payments between the original bank and the second bank made for consumers' payments to entrepreneurs and entrepreneurs' repayment of loans. However, there are zero net end-of-day settlements actually paid between banks. Correspondingly, the average daily transactions made over CHIPS, the private New York-based clearinghouse which uses a netting system, was \$1.2 trillion. Of this, net end-of-day settlements paid were only \$7 billion (Henckel et al., 1999). Thus, there was a turnover ratio of 171 for every net dollar paid between banks over CHIPS.

Finally, the amount of interbank lending that may potentially occur in the model relates to the amount of actual interbank lending that occurs in the U.S. The average overnight federal

[6] I assume in my model either that there is no charge and no maximum for negative intraday balances, or that the level of negative balances for the bank within my model never reaches the imposed limit. This could be because the limit on negative balances is very large, or because the bank borrows throughout the day to maintain a zero balance.

funds activity (uncollateralized interbank lending in the U.S.) in 1998 is estimated to be \$144 billion (Furfine, 1999), while the average overnight bank balances held at the Federal Reserve in August 2001 is \$15 billion (McAndrews and Potter, 2002). This implies that nearly 90% of funds borrowed by banks are relent within the same day and in net are not held as overnight borrowing.

This paper also relates to the literature on money. In order to introduce money into the economy, the central bank establishes the unit of account value of money under a "gold standard" by committing to buy or sell goods for fiat money at a fixed price. Money continues to have value due to its payments role as numeraire in a credit and debit economy after the central bank drops its backing for money. This is similar to the payments value of money that Diamond and Rajan (2003b) use.

Specific frictions not examined in this paper may give rise to bank runs even with efficient interbank lending. Holmström and Tirole (1998) and Diamond and Rajan (2003a) show that when banks cannot fully collect from entrepreneurs, either banks or entrepreneurs may not be able to borrow against the value of their future loans and are susceptible to liquidity runs and insolvency. If banks experience individual losses on loans, insolvency would lead to bank runs and perhaps systemic risk, as shown in Rochet and Tirole (1996) and Aghion et al. (2000).

3. The Benchmark Model

The framework of my model is similar to that which has become standard in the literature since Diamond and Dybvig (1983) but with the addition of money and entrepreneurs. There are three periods, $t = 0, 1, 2$. Consumers are endowed with goods at $t = 0$. The fraction λ of consumers receive an unverifiable liquidity shock and need to consume at $t = 1$, where $0 < \lambda < 1$. These early consumers have utility given by $U = u(C_1)$. The fraction $1 - \lambda$ of consumers do not receive a liquidity shock and consume at $t = 2$. These late consumers have utility $U = u(C_2)$. Period utility functions $u(\cdot)$ are assumed to be twice continuously differentiable, increasing, strictly concave and satisfy Inada conditions $u'(0) = \infty$ and $u'(\infty) = 0$. Consumers do not know their type at $t = 0$. I assume there is a large finite number of consumers, and I normalize the number of consumers and the amount of goods held by consumers at $t = 0$ to one. Goods are storable over a period.

Banks are competitive and take deposits from consumers and lend to short term and long term entrepreneurs, who store or invest goods. Goods invested at $t = 0$ return $R > 1$ at $t = 2$, or alternatively return $r < 1$ if the investment is liquidated at $t = 1$. Entrepreneurs have no endowment and are risk neutral, competitive and maximize profits in terms of final goods they hold and consume at $t = 2$. Without loss of generality, I treat all short term entrepreneurs as a single short term entrepreneur (he), and all long term entrepreneurs as a single long term

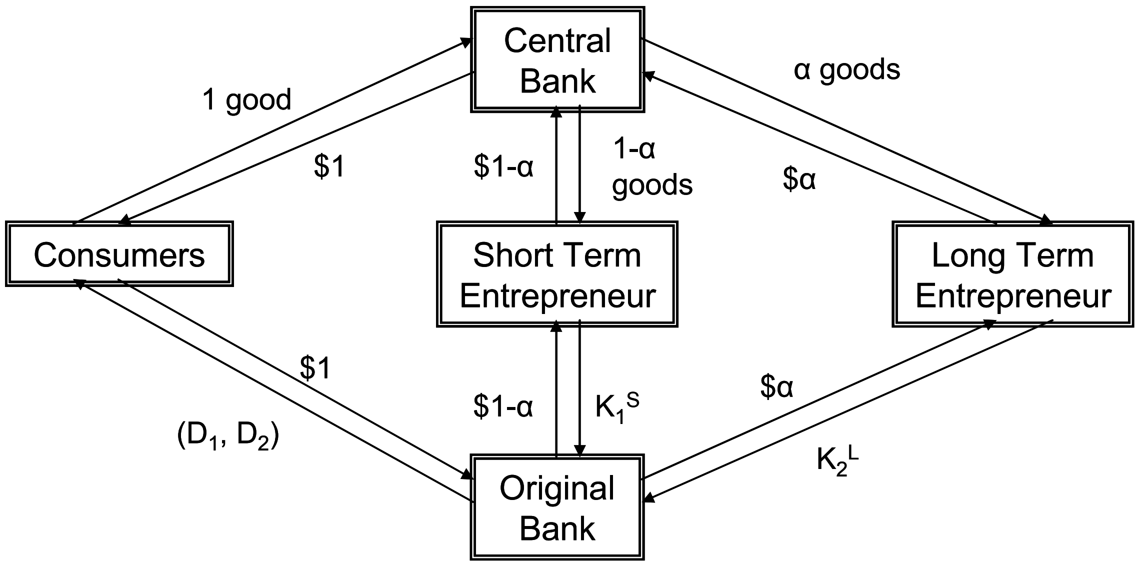

Figure 3.1: Introduction of Money and Nominal Contracts at $t = 0$

entrepreneur (she), both who are price takers.

At $t = 0$, the central bank creates fiat money for initial exchange under a "gold standard" in which it buys and sells goods at a fixed price, but it receives all the money back at $t = 0$ and plays no role in the benchmark model thereafter. This establishes money as the unit of account and determines prices at $t = 0$, which then carries over to later periods due to the system of credits and debits created at $t = 0$ throughout the economy, even though the ongoing net supply of money is zero.

Timeline Figure 3.1 illustrates the introduction of money and nominal contracts at $t = 0$. Consumers sell their goods to the central bank for dollars at the price set by the central bank, $P_0 = 1$. The consumers deposit their money in the original bank in exchange for a demand deposit account (D_1, D_2), where either D_1 or D_2 is the amount of money payable at $t = 1$ or $t = 2$, respectively, per unit of deposit.[7,8] For uniformity, variables are denoted by the subscript "t" for the time period and by the superscript "i" for the agent.

The bank lends $(1 - \alpha)$ dollars to the short term entrepreneur for a debt contract of K_1^S, due at $t = 1$. The bank lends α dollars to the long term entrepreneur for a debt contract of K_2^L, due at $t = 2$. The short term entrepreneur buys and stores β_0^S goods from the central

[7] Banks are mutually owned and consumers who withdraw D_2 at $t = 2$ are also the residual claimants on the bank after bank claims are paid at $t = 2$.

[8] Throughout the paper, demand deposit contracts refer to quantities per unit of money deposited and consumption refers to quantities per unit-sized consumer unless otherwise specified.

bank at a price of $P_0 = 1$. The long term entrepreneur buys and invests α goods from the central bank at a price of $P_0 = 1$.[9,10]

At the end of this exchange at $t = 0$, the net holdings are as follows. The central bank does not hold any money or goods. The consumer holds the demand deposit account (D_1, D_2), the short term entrepreneur holds $1 - \alpha$ in stored goods, the long-term entrepreneur holds α in the long term investment, and the bank holds the debt contracts K_1^S and K_2^L due from the entrepreneurs. The bank does not hold reserves in the benchmark model since this is not the focus.[11]

A second bank represents the interbank market and allows for deposit accounts and payments outside of the original bank. Without loss of generality, the efficient interbank lending market is modeled as this single bank which does not have deposits or loans at $t = 0$.[12] After $t = 0$, all money is transferred electronically among the accounts of consumers, entrepreneurs and the banks own accounts. For clarity, I always refer to the "second bank" as such, and refer to the "original bank" as such or simply as the "bank."

Figure 3.2 illustrates the transactions that take place at $t = 1$. The dashed arrows represent out-of-equilibrium actions. At $t = 1$, $\lambda^w \leq 1$ fraction of consumers withdraw D_1 from the bank. The fraction $\lambda^p \leq \lambda^w$ of consumers purchase goods from either entrepreneur at the market clearing price P_1. The remaining $\lambda^w - \lambda^p$ consumers who have withdrawn transfer their funds to the second bank for a one-period demand deposit contract $D_{1,2}$ payable by the second bank at $t = 2$. Since early consumers must consume at $t = 1$, $\lambda^p \geq \lambda$. A bank run is defined as when any late consumers withdraw at $t = 1$ to either purchase goods or redeposit: $\lambda^w > \lambda$.

The short term entrepreneur may choose to store β_1^S goods until $t = 2$, leaving him

$$Q_1^S \equiv \beta_0^S - \beta_1^S$$

to sell at $t = 1$. The short term entrepreneur then repays the bank his debt contract. The long

[9] Since entrepreneurs break even in equilibrium, they always accept the loan. Entrepreneur would not choose to borrow money if not to buy goods at $t = 0$.

[10] Money and nominal contracts could be introduced without the central bank exchanging goods. Consumers could deposit goods at the bank for (D_1, D_2) and the bank could lend the goods to entrepreneurs for K_1^S and K_2^L. However, at $t = 0$ the central bank must offer to exchange money for goods at P_0 in order to establish the unit of account and determine prices.

[11] This corresponds closely to the decreasing amount of reserves held by banks in reality, shown by Woodford (2000). Moreover, the results show no bank runs even without reserves held. The model is expanded to include reserves in Section 6 where currency is added.

[12] If there were multiple banks that all held deposits and illiquid investments at the beginning of the timeline, the results of the benchmark model would be unchanged. Any money withdrawn for purchases or redeposits at any bank would be available for lending from whichever banks received the money.

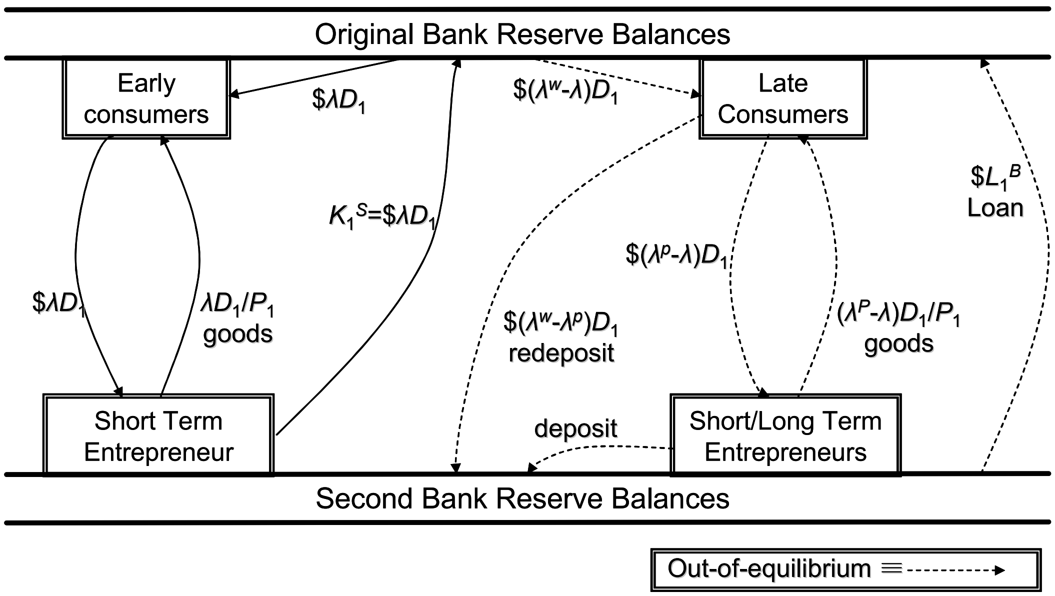

Figure 3.2: Transactions at $t = 1$

term entrepreneur may choose to liquidate γ_1^L of her invested goods, giving her

$$Q_1^L \equiv \gamma_1^L r$$

goods to sell at $t = 1$. The short and long term entrepreneurs store all electronic funds received in demand deposit accounts at the second bank that pay $D_{1,2}$ at $t = 2$ as well. Both the original and second banks offer the same rate $D_{1,2}$ on demand deposits at $t = 1$, but I assume entrepreneurs deposit at the second bank to allow for greater interbank lending. The second bank may lend L_1^B to the original bank at $t = 1$ for a gross rate of return of $D_{1,2}^{ff}$ (corresponding to the federal funds rate in the U.S.) due at $t = 2$.

Figure 3.3 illustrates the transactions that occur at $t = 2$. At $t = 2$, the bank pays D_2 to the $1 - \lambda^w$ fraction of late consumers who arrive for late withdrawal. The second bank pays a return rate of $D_{1,2}$ to late consumers (and entrepreneurs) who have deposits there. These $1-\lambda^p$ late consumers who withdraw and have not purchased goods at $t = 1$ now purchase goods from the long term entrepreneur at the market clearing price P_2. The long term entrepreneur sells Q_2^L goods at $t = 2$. She repays her debt contract to the bank and consumes any excess goods held as profit. The short term entrepreneur never sells goods at $t = 2$ since he has no debts to repay at $t = 2$. He may purchase goods at $t = 2$ if he has money left over after repaying K_1^S and consumes any goods held as profit. The original bank repays its loan of $L_1^B D_{1,2}^{ff}$ to the second bank.

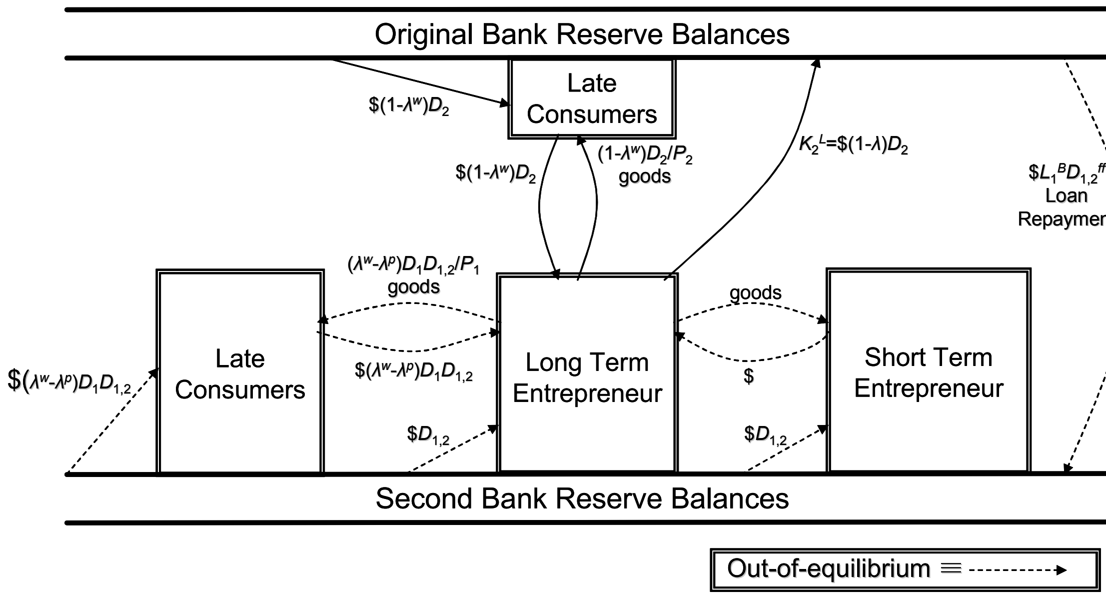

Figure 3.3: Transactions at $t = 2$

Assumptions All payments at $t = 1$ and $t = 2$ are made in dollars electronically paid between banks within a clearinghouse under a netting system. Each bank's budget constraint is given by a same-period payment-in-advance constraint: a bank can make any amount of payments during a period (and so carry a negative intraperiod balance) provided at the end of the period all payments made and received net to a nonnegative balance, otherwise the bank defaults. If either bank cannot pay its depositors and loan repayment at either period in full, it defaults. Interbank loans have a junior claim to demand deposits. This is necessary so that the second bank cannot expropriate late consumers who do not withdraw at $t = 1$ when it lends to the original bank, and so it is naturally a clause in the demand deposit contract that the original bank includes at $t = 0$ in order to maximize $t = 0$ depositors' welfare. Among depositors, I allow for any of the following bank default rules to be in effect. A pro-rata rule specifies that all withdrawing depositors receive evenly divided proceeds. A sequential service rule specifies that depositors receive their demand deposit claims in full according to the order of their withdrawal requests until the bank defaults.[13] A callable-loan rule specifies that if

[13]Diamond and Dybvig (1983) model the sequential service constraint by assuming consumer withdrawals are ordered in time within a period in their model and paid according to the order. They use this to try to reflect continuous time withdrawals within a discrete time model. Allen and Gale (1998) argue that the description by Diamond and Dybvig (1983) is in opposition to the historical application of the sequential service constraint during bank runs. During periods without runs, consumers are in practice paid sequentially on a day-by-day basis. Allen and Gale (1998) point out that during a bank run, all consumers attempt to withdraw on the same day, and within the period of a day consumer withdrawals are not treated sequentially but rather on a pro-rata

the bank were going to default, it recalls the loan to the long term entrepreneur in sufficient quantity until default is prevented. If there is not a callable-loan rule, at $t = 2$ all unfulfilled claims from $t = 1$ must be paid in full before $t = 2$ claims are paid. If an entrepreneur can not repay his/her debt in full, the entrepreneur defaults and must sell all goods possessed at the market price in the period the debt is due and pay all proceeds toward the debt repayment.

Banks can ensure that at $t = 0$ short term entrepreneurs only store goods and long term entrepreneurs only invest goods. This is an important assumption because a key function of a bank is to ensure the proper amounts of storage and investment. In fact, I show in Lemma 1 in the next section that the market would typically provide first best consumption without banks if all consumers individually stored and invested optimal amounts at $t = 0$ and then traded. The bank provides this function by ensuring optimal storage and investment at $t = 0$, but entrepreneurs (the market) are free to choose how much to store, liquidate and sell after $t = 0$. Including entrepreneurs who borrow and invest is more realistic than assuming physical investment by banks and allows for studying the interplay of bank financing to firms and firms' real investment.[14] This also allows for distinguishing the bank's role of providing consumers with liquidity insurance by paying fixed demand deposits, from the market's role of allocating goods according to the price mechanism, which inherently is efficient once the optimal amount is stored and invested.

For simplification of the model, I assume the entrepreneurs cannot receive additional loans from the second bank at $t = 1$ or renegotiate their loans with the original bank. These assumptions do not change the results of the benchmark model and in fact strengthen the robustness of the no-bank runs results. Allowing for additional loans or renegotiation would simply help entrepreneurs react to potential depositor runs with more flexibility in repaying loans to the original bank and in providing consumers optimal consumption at $t = 1$ and $t = 2$, both of which would ensure against runs even further. If fact, if entrepreneurs could borrow from additional banks at $t = 2$, bank runs would not occur even when interbank lending is inefficient and breaks down, as in Section 7 on banking crises below. Moreover, partial renegotiation is assumed in that section in order to limit bank runs which occur due to inefficient interbank lending.[15]

basis. A sequential service constraint is not necessary to produce a bank run in a real contracts model. A bank run can exist under a pro-rata rule if paying the demand deposits under a run causes sufficient liquidation of goods that late consumers prefer to run.

[14] See Diamond and Rajan (2001, 2003a, 2003b) for a further developed model on bank lending to entrepreneurs.

[15] The economic justification for no additional loans at $t = 1$ is that the original bank has established a banking relationship with the entrepreneur which enables it to collect on its loan, so the second bank cannot also establish a banking relationship to collect on loans with the same entrepreneur. The justification for no renegotiation is that the mutual bank is established among depositors at $t = 0$ as a set of contracts, which includes the loans to entrepreneurs made at $t = 0$. Because of the difficulty of renegotiating with the numerous

4. No Bank Runs with Nominal Contracts

4.1. First Best Solution

The first best allocation is what a benevolent social planner would provide based on observing consumer types. The first best allocation in my model is the same as in the model of Diamond and Dybvig (1983) and maximizes the consumer's expected utility:

$$\max_{C_1,C_2,\alpha} \quad \lambda u(C_1) + (1-\lambda)\, u(C_2) \tag{4.1a}$$

$$\text{s.t.} \quad \lambda C_1 \leq 1-\alpha \tag{4.1b}$$

$$(1-\lambda)\, C_2 \leq \alpha R. \tag{4.1c}$$

Since λ is known, optimal consumption requires that early consumers only consume from goods stored at $t=0$ (4.1b), and late consumers only consume from goods invested at $t=0$ (4.1c). This ensures no inefficient liquidation and no underinvestment of goods. The first-order conditions and binding constraints give the implicit first best solution C_1^*, C_2^* and α^*, according to

$$\frac{u'(C_1^*)}{u'(C_2^*)} = R \tag{4.2a}$$

$$\lambda C_1^* = 1-\alpha^* \tag{4.2b}$$

$$(1-\lambda)\, C_2^* = \alpha^* R. \tag{4.2c}$$

(4.2a) shows that the ratio of marginal utilities between $t=1$ and $t=2$ is equal to the marginal rate of transformation R. Since $\lambda > 0$, $\alpha^* < 1$.

4.2. Market Solution

In a market without banks or entrepreneurs, in which consumers store and invest goods themselves, the well known outcome is $C_1 = 1$, $C_2 = R$. Consumers each invest $\alpha = 1-\lambda$, and then early consumers trade their investment of α for the storage of late consumers $1-\lambda$ at the price of one current good for one invested good at $t=1$. This is not first best (except for the special case of log utility). The reason is that insurance cannot be provided through contracting since types are not verifiable, and consumers cannot ex-ante commit to not be ex-post opportunistic and trade at the spot price for early verses late goods at $t=1$. Therefore, consumers do not store and invest optimally at $t=0$.

If consumers could be forced to store $1-\alpha^*$ and invest α^*, the market achieves the first best

depositors, the bank cannot reoptimize its contracts at $t=1$ and the loans are not renegotiable.

outcome if consumers have the typically assumed coefficient of relative risk aversion (CRRA) $\frac{-cu''(c)}{u'(c)}$ greater than one.[16] Early consumers would trade α^* invested goods for $1-\alpha^*$ stored goods from late consumers at $t=1$. This is important because it indicates that a key role of a bank is to ensure optimal storage and investment. Moreover, it gives insight into the ability of the market to provide optimal consumption for consumers once the optimal storage and investment is made.

Lemma 1. *The unique market equilibrium when consumers are required to store $1-\alpha^*$ and invest α^* (for CRRA greater than one) is the first best outcome $C_1 = C_1^*$ and $C_2 = C_2^*$.*

Proof. See Appendix. ■

The bank solution works below even for CRRA less than one because once a consumer discovers he is an early type, his deposit is already made and he cannot refuse to participate.

4.3. Banking Solution with Real-Goods Contracts

In the standard Diamond-Dybvig model with no money, no entrepreneurs, and demand deposits payable in real goods, a bank can provide the first best solution through demand deposits in which consumers deposit their goods at $t=1$ and simply show up at either $t=1$ or $t=2$ to receive C_1^* or C_2^*, respectively. Since the first-order condition $u'(C_1^*) = Ru'(C_2^*)$ implies $C_2^* > C_1^*$, the incentive constraint for consumers is satisfied. A bank can provide the first best allocation without observing the individuals' types because demand deposits commit the bank at $t=0$ to store and invest optimally or else it cannot repay depositors at either $t=1$ or $t=2$ and would default.

The problem with bank demand deposits is that the first best equilibrium is not unique. If at $t=1$ each consumer believes that all other consumers are going to withdraw from the bank, it is a self-fulfilling prophecy and a bank run is another possible equilibrium. All consumers withdraw at $t=1$, forcing liquidation of investments and sub-optimal consumption.

4.4. Banking Solution with Nominal Contracts

Now I turn to the model with nominal contracts and money. In order to solve the model, I first conjecture the contracts that are offered by the bank and claim they maximize consumers' utility. I then solve for optimal behavior by the entrepreneurs, second bank, and consumers.

[16] This is assumed by much of the banking literature starting with Diamond and Dybvig (1983) and implies that banks perform a risk-decreasing consumption insurance role for early consumers, where $C_1^* > 1$, so that banks provide greater consumption for early consumers than the market would. If the coefficient of relative risk aversion is less than one, then $C_1^* < 1$ and $C_2^* > R$, so the bank acts as a risk-increasing gamble in which early consumers receive lower consumption and late consumers receive greater consumption than that provided by the market.

I show that the outcome is a unique equilibrium with first best results, confirming the banks decision.

Bank Contracts At $t = 0$, the bank offers consumers the demand deposit contract (D_1, D_2), lends $1 - \alpha$ to the short term entrepreneur for the debt contract K_1^S, and lends α to the long term entrepreneur the debt contract K_2^L, where

$$
\begin{aligned}
D_1 &= C_1^* \\
D_2 &= C_2^* \\
K_1^S &= 1 - \alpha = \lambda D_1 && (4.3) \\
K_2^L &= \alpha R = (1 - \lambda) D_2 && (4.4) \\
\alpha &= \alpha^*.
\end{aligned}
$$

The second equality in (4.3) and in (4.4) holds due to the the first-order conditions (4.2b) and (4.2c) above. The consumer's incentive constraint holds since $D_2 > D_1$. What is required is to show the unique equilibrium

$$
\begin{aligned}
\lambda^w &= \lambda^p = \lambda && (4.5) \\
P_1 &= P_2 = 1, && (4.6)
\end{aligned}
$$

which means that there are no bank runs and prices reflect the optimal allocation of goods. Then consumption is given by

$$
\begin{aligned}
C_1 &= \frac{D_1}{P_1} = C_1^* \\
C_2 &= \frac{D_2}{P_2} = C_2^*.
\end{aligned}
$$

Since this satisfies the consumer's incentive constraint $C_2 \geq C_1$, the first best results obtain as the unique equilibrium.

A first best outcome requires that the short term entrepreneur chooses to sell his entire stock of goods at $t = 1$,

$$\beta_1^S = 0, \tag{4.7}$$

and the long term entrepreneur chooses to not liquidate any invested goods at $t = 1$ and to sell all of her goods at $t = 2$,

$$
\begin{aligned}
\gamma_1^L &= 0 && (4.8) \\
Q_2^L &= \alpha R. && (4.9)
\end{aligned}
$$

Entrepreneur Optimizations The long-term entrepreneur's optimization problem at $t = 1$ is to maximize her total consumption of goods, given as follows:

$$\max_{Q_2^L, \beta_1^L, \gamma_1^L} \quad [\overline{Q}_2^L(\gamma_1^L, \beta_1^L) - Q_2^L \mid \lambda^w, \lambda^p] \tag{4.10a}$$

$$\text{s.t.} \quad \widetilde{Q}_2^L \leq Q_2^L \tag{4.10b}$$

$$Q_2^L \leq \overline{Q}_2^L \tag{4.10c}$$

$$\gamma_1^L \leq \alpha \tag{4.10d}$$

$$\beta_1^L \leq \gamma_1^L r \tag{4.10e}$$

with the requirement that Q_2^L, β_1^L and γ_1^L are nonnegative. (4.10b) is the constraint on the goods to be sold, Q_2^L, at $t = 2$, to satisfy the long term entrepreneur's outstanding debt constraint expressed in real terms, where $\widetilde{Q}_2^L \equiv \min\left\{\frac{M_2^L}{P_2}, \overline{Q}_2^L\right\}$. $M_2^L \equiv K_2^L - Q_1^L P_1 D_{1,2}$ is the quantity of money demanded by the long term entrepreneur to repay her loan. If she defaults, she must sell all of her goods, defined as $\overline{Q}_2^L \equiv (\alpha - \gamma_1^L)R + \beta_1^L$, where β_1^L is the goods stored by the long term entrepreneur at $t = 1$. This is clearly zero since she would never liquidate investments to store goods. (4.10c) is the constraint on the amount of goods to be sold, Q_2^L, at $t = 2$, based on available quantity. (4.10d) is the constraint on the total goods available for liquidation, γ_1^L, at $t = 1$. (4.10e) is the constraint on goods available for storage, β_1^L, at $t = 1$.

The short-term entrepreneur's optimization problem at $t = 1$ is to maximize his total consumption of goods, given as follows:

$$\max_{\beta_1^S} \quad \left[\beta_1^S + \frac{S_2^S(\beta_0^S, \beta_1^S) D_{1,2}}{P_2} \mid \lambda^w, \lambda^p\right] \tag{4.11a}$$

$$\text{s.t.} \quad \widetilde{Q}_1^S \leq Q_1^S \tag{4.11b}$$

$$\beta_1^S \leq \beta_0^S \tag{4.11c}$$

with the requirement that β_1^S is nonnegative. $S_2^S \equiv (Q_1^S P_1 - K_1^S)^+$ is the quantity of money the short term entrepreneur deposits at $t = 1$ to use to buy goods at $t = 2$. (4.11b) is the constraint on the minimum amount of goods Q_1^S that must be sold at $t = 1$ to satisfy his debt constraint expressed in real terms, where $Q_1^S \equiv \min\left\{\frac{K_1^S}{P_1}, \beta_0^S\right\}$. If he defaults, he must sell β_0^S. (4.11c) is the constraint on goods available for storage, β_1^S, at $t = 1$.

Prices are determined at $t = 1$ and $t = 2$, according to market clearing conditions, as the amount of money supplied for purchasing goods in a period divided by the amount of goods supplied for sale as follows:

$$P_1 \equiv \frac{\lambda^p D_1}{Q_1^S + Q_1^L}$$
$$P_2 \equiv \frac{S_2}{Q_2^L},$$

where

$$S_2 \equiv (1-\lambda^w)D_2 + (\lambda^w - \lambda^p)D_1 D_{1,2} + S_2^S D_{1,2}$$

is the total quantity of money supplied by late consumers and the short term entrepreneur to buy goods at $t = 2$.

Consumer Optimizations At each period $t = 1, 2$, consumers can choose to withdraw from the bank by either redepositing funds at the second bank or by submitting a demand schedule for purchase of goods. At $t = 1$, early consumers always fully withdraw and purchase goods at the market price. Late consumers choose whether to withdraw early or not and, if so, whether to purchase goods at $t = 1$ or redeposit and purchase goods at $t = 2$ in order to maximize consumption. I assume that if late consumers are indifferent between withdrawing or not withdrawing at $t = 1$, they choose not to withdraw. This assumption is to simplify semantics and notation only. This assumption is used to ensure that late consumers must strictly prefer to withdraw in order for those withdrawals to be considered a bank run, as is the case in the banking literature (see for instance Diamond and Dybvig, 1983, and Allen and Gale, 1998).[17]

If the original bank is not expected to default, late consumers do not withdraw and redeposit at the second bank at $t = 1$ because the second bank can never pay more than the original bank: $D_1 D_{1,2} \le D_2$. Late consumers do withdraw and buy goods at $t = 1$ if their consumption from this strategy, $\frac{D_1}{P_1}$, is greater than their consumption from withdrawing and buying goods at $t = 2$, $\frac{D_2}{P_2}$. Thus, if $\frac{D_1}{P_1} > \frac{D_2}{P_2}$, late consumers choose to withdraw and purchase goods early at $t = 1$, so $\lambda^w = \lambda^p = 1$ (the bank is run). If $\frac{D_2}{P_2} \ge \frac{D_1}{P_1}$, late consumers choose to withdraw late at $t = 2$, so $\lambda^w = \lambda^p = \lambda$. Intermediate cases of $\lambda < \lambda^p < 1$ require that consumption from late withdrawal is equivalent to early withdrawal and purchase: $\frac{D_2}{P_2} = \frac{D_1}{P_1}$.

If the bank were to default at $t = 2$, the bank's repayment on its loan to the second bank and possibly on late withdrawals would be reduced, but payment on early withdrawals is already made and would not be reduced. If the bank defaults at $t = 1$, since early withdrawals

[17] If late consumers were to withdraw and redeposit or pay money to entrepreneurs who deposit it at the second bank when indifferent, the original bank could always borrow and repay the funds from the second bank. The bank would not default and the outcome would be first best. This type of withdrawal would be inconsequential and would not be properly described as a bank "run". Rather it would simply be a bank withdrawal, so the mathematical definition of a bank run as $\lambda^w > \lambda$ would have to be redefined in accordance and there would still be a unique equilibria without bank runs.

are senior to late withdrawals, the bank's payment on D_1 is never reduced unless payment on D_2 is zero. Thus, if at $t = 1$, any or all late consumers expect the bank to default in either period, there may be more late consumers who withdraw at $t = 1$ than when late consumers do not expect the bank to default. This implies that λ^w and λ^p may be greater but not less for an expected bank default than for no expected bank default, given $\frac{P_2}{P_1}$.

Interbank Lending The interbank loan required at $t = 1$ by the original bank,

$$L_1^B \equiv \lambda^w D_1 - \min\{K_1^S, Q_1^S P_1\},$$

is the difference between the amount of money paid on withdrawals at $t = 1$ and the amount of money received from the short term entrepreneur, which depends on whether he defaults on K_1^S. The funds that the second bank has available to lend,

$$(\lambda^w - \lambda^p)D_1 + S_2 + Q_1^L P_1,$$

are equal to the loan required by the bank since the money paid by the original bank is deposited in consumer or entrepreneur accounts at the second bank.

I make the assumption of efficient interbank lending, which means that the second bank lends if the original bank will not default. The gross interest rate on the loan may be any feasible rate $D_{1,2}^{ff} \geq 1$ such that it does not cause the original bank to default. If late consumers simply redeposit money at the second bank, the original bank can borrow and repay the funds to the second bank at an interest rate up to the return $\frac{D_2}{D_1}$ that the original bank would have paid had the late consumers withdrawn at $t = 2$ instead. However, if late consumers buy goods at $t = 1$, the original bank can only repay a loan from the second bank at $t = 2$ if it will receive enough from the late entrepreneur to repay depositors and the loan and $t = 2$:

$$\min\{K_2^L, Q_2^L P_2 + Q_1^L P_1 D_{1,2}\} \geq L_1^B D_{1,2}^{ff} + (1 - \lambda^w)D_2 = L_1^B D_{1,2} + (1 - \lambda^w)D_2. \qquad (4.12)$$

This is a more subtle requirement that the following lemma shows does hold for all λ^w and λ^p—even if there is a run. This is true because the original bank pays out a maximum of $\lambda D_1 + (1 - \lambda) D_2$ under any scenario of potential depositor behavior. The rate paid on the loan is $D_{1,2}^{ff} \leq \frac{D_2}{D_1}$, shown in the lemma. This implies that when late consumers withdraw early, the original bank pays no more than $D_1 D_{1,2}^{ff} \leq D_2$ on the withdrawal or repayment on the loan per late consumer who withdraws. The maximum the bank must pay of $\lambda D_1 + (1 - \lambda) D_2$ equals the total of the loan payments due by the entrepreneurs, $K_1^S + K_2^L$. Thus, (4.12) implies the bank must simply meet a two-period budget constraint. Competition among entrepreneurs for money to repay loans ensures that money supplied to buy goods is spread efficiently among entrepreneurs, and competition in the goods market limits profit taking by entrepreneurs, regardless of λ^p and λ^w. For example, if $\lambda^p > \lambda$, so that more money is spent on goods at $t = 1$, P_1 increases and P_2 decreases. The long term entrepreneur may sell goods at $t = 1$ to capture revenues that are lost from $t = 2$ sales. However, if r is so low that the long term entrepreneur cannot liquidate enough investments and sell enough goods at $t = 1$, the short term entrepreneur will receive revenues greater than K_1^S. Since the long term entrepreneur has a greater demand for the excess revenues than the short term entrepreneur, the short term entrepreneur will buy goods from the long term entrepreneur at $t = 2$. This implies that the short term entrepreneur does make a profit in goods consumed, but it is at the expense of consumers consuming less. Entrepreneurs repay loans in adequate amounts that the original bank does not default.

Since the second bank is competitive with a zero profit condition, it pays the entire return from the loan $L_1^B D_{1,2}^{ff}$ on the demand deposit accounts to consumers and entrepreneurs. If the second bank lends all of its funds from consumers and entrepreneurs at $t = 1$ to the original bank, the second bank's demand deposit contract to consumers and entrepreneurs is $D_{1,2} = D_{1,2}^{ff}$.

Lemma 2. *The original bank always receives any needed loan L_1^B from the second bank and never defaults on the loan repayment or depositor withdrawals at $t = 1$ or $t = 2$, for all λ^w and λ^p. The return on the loan and on deposits made at $t = 1$ are $D_{1,2} = D_{1,2}^{ff} \in \left[1, \frac{D_2}{D_1}\right]$.*

Proof. See Appendix. ■

Goods Market Though the bank never defaults, if any late consumers run the bank and buy goods at $t = 1$, there cannot be efficient consumption. In order to prove a first best outcome, I still need to show that there are no actual bank runs. In order to do this, I solve for the entrepreneurs' choices over the quantity of goods to sell each period, incorporating the late consumers actions as a function of prices. This then determines market prices and consumer actions.

First, consider the case that late consumers withdraw and purchase goods at $t = 2$. If the short term entrepreneur sells all his goods at $t = 1$ and the long term entrepreneur sells all her goods at $t = 2$, $P_1 = P_2 = 1$. Instead, if the long term entrepreneur were to sell some goods at $t = 1$, an increase in goods at $t = 1$ implies $P_2 < 1$, and a decrease in goods at $t = 2$ implies $P_2 > 1$, so P_2 would be greater than P_1. Thus, the long term entrepreneur prefers to sell all goods only at $t = 2$. Rather than this, if the short term entrepreneur were to sell some goods at $t = 2$, then $P_1 > P_2$. Thus, the short term entrepreneur prefers to sell all goods at $t = 1$. Finally, if instead the long term entrepreneur were to sell goods at $t = 2$ and the short term entrepreneur were to sell goods at $t = 1$ simultaneously, fewer total goods could be sold since the long term entrepreneur would not receive the return of $R > 1$ on all her goods. So the long term entrepreneur would always prefer to sell all goods at $t = 2$ unless $P_1 > P_2$, in which case the short term entrepreneur would prefer to sell all goods at $t = 1$. Thus, the long (short) term entrepreneur prefers to sell goods only at $t = 2$ ($t = 1$), and the goods market provides the optimal allocation of goods to early and late consumers.

Next, consider the case that some or all late consumers run the bank by withdrawing early and purchasing goods or redepositing with the second bank. Even with an optimal response of some possible liquidation and sale of goods by the long term entrepreneur at $t = 1$, P_1 either rises or is unchanged and P_2 either falls or is unchanged due to increased consumer demand at $t = 1$. Furthermore, late consumers receive less money by withdrawing at $t = 1$ than at $t = 2$, because the bank does not default, as shown by Lemma 2. Withdrawing at $t = 1$ pays $D_1 < D_2$. Withdrawing and redepositing pays $D_1 D_{1,2} \leq D_2$. Thus, the late consumers' optimization shows that late withdrawal gives greater consumption than early withdrawal and purchase $\left(\frac{D_2}{P_2} > \frac{D_1}{P_1}\right)$ or than early withdrawal, redeposit and late purchase $\left(\frac{D_2}{P_2} \geq \frac{D_1 D_{1,2}}{P_1}\right)$.

The important point is that the marginal late consumer prefers to withdraw at $t = 2$ even if other late consumers are running the bank. Even during a bank run, the goods market provides any late withdrawing consumer his optimal allocation of goods at a minimum. Hence, no late consumers choose to run, and anticipated bank runs do not materialize. The difference from unavoidable bank runs in Diamond and Dybvig (1983) is that with real-goods deposits, the bank pays fixed amounts of goods until it runs out. With nominal contracts, the bank never runs out of money to borrow to pay fixed deposits, and the goods market rations consumption efficiently to consumers through the market price mechanism.

Proposition 1. *The unique equilibrium of the benchmark model with nominal contracts is the first best outcome* $C_1(P_1) = \frac{D_1}{P_1} = C_1^*$, $C_2(P_2) = \frac{D_2}{P_2} = C_2^*$ *and* $\alpha = \alpha^*$, *with no bank runs.*

Proof. First, I solve the short and long term entrepreneurs' optimizations to show that the unique solution is (4.7) and (4.8). (See Appendix).

Second, I show that at $t = 2$, the long term entrepreneur sells all of his goods, $Q_2^L = \overline{Q}_2^L$. Suppose not, $Q_2^L \neq \overline{Q}_2^L$. This implies $S_2 = M_2^L = 0$ and $Q_2^L = 0$ from Lemma 2.2, which is given in the appendix. But $Q_1^L = 0$ implies $M_2^L = K_2^L > 0$, a contradiction. Thus, $Q_2^L = \overline{Q}_2^L$.

Third, I show that all late consumers withdraw and purchase goods at $t = 2$, $\lambda^w = \lambda^p = \lambda$. By Lemma 2, the bank never defaults. So by the late consumers' optimization, they never withdraw at $t = 1$ and redeposit, $\lambda^w - \lambda^p = 0$. If $\frac{D_2}{P_2} > \frac{D_1}{P_1}$, the late consumers withdraw at $t = 2$. I will show that $P_1 \geq P_2$. Suppose not, $P_1 < P_2$. Suppose also $S_2^S > 0$. This implies $P_1 > 1$. However, this implies $S_2 = Q_2^L P_2 > \overline{Q}_2^L = \alpha R$, and $M_2^L = K_2^L = \alpha R$, so $S_2 > M_2^L$, a contradiction to Lemma 2.2. Thus, since $S_2^S \geq 0$, $S_2^S = 0$. Substituting for P_1 and P_2, $P_1 < P_2$ implies $\frac{\lambda^p D_1}{\lambda D_1} < \frac{(1-\lambda^p) D_2}{(1-\lambda) D_2}$, or $\lambda^p < \lambda$, a contradiction. Therefore, $P_1 \geq P_2$, which implies $\frac{D_2}{P_2} > \frac{D_1}{P_1}$, since $D_2 > D_1$. This implies by the late consumers' optimization that $\lambda^w = \lambda^p = \lambda$.

Finally, I show that the equilibrium is unique and the outcome is first best. Since the entrepreneurs' choices, (4.7), (4.8) (4.9), and consumers' choices, (4.5), are uniquely determined, prices $P_1 = \frac{\lambda^p D_1}{Q_1^S + Q_1^L} = 1$ and $P_2 = \frac{S_2}{Q_2^L} = 1$ are uniquely determined. Thus, $C_1 = \frac{D_1}{P_1} = C_1^*$ and $C_2 = \frac{D_2}{P_2} = C_2^*$ are uniquely determined, and the consumer's incentive constraint, $C_2 \geq C_1$, is satisfied. Moreover, this is the first best outcome. This confirms the conjecture of the bank's choice of contracts offered. Thus, $C_1 = C_1^*$, $C_2 = C_2^*$ and $\alpha = \alpha^*$ is the unique equilibrium. ■

5. No Bank Runs with Aggregate Uncertainty of Returns

Allen and Gale (1998) argue that bank runs occur due to the aggregate uncertainty of investment returns, which correspond to the macroeconomic business cycle. I extend the benchmark model with nominal contracts, money and entrepreneurs to the Allen and Gale (1998) framework of aggregate uncertainty of returns. The differences in the Allen and Gale (1998) framework from the framework based on Diamond and Dybvig (1983) are due to the return of the long term investment. The long term investment return R is random with a density function $f(R)$, where $R \geq 0$. At $t = 1, R$ is observable to everyone but not verifiable so cannot be contracted upon. In addition, the long term investment cannot be liquidated at $t = 1$ to recover any goods, $\gamma_1^L = 0$. The return on goods stored by the bank (stored by the entrepreneur in my model) between $t = 1$ and $t = 2$ is $\rho \geq 1$, where consumers can store goods for a return of only one between $t = 1$ and $t = 2$. The reason that ρ is introduced is to allow banks (or entrepreneurs in my model) to store goods more efficiently than consumers. The return on goods stored by any party between $t = 0$ and $t = 1$ is one. Technical assumptions to ensure interior solutions are

$$\begin{aligned} E[R] &> 1 \\ u'(0) &> E[u'(R)R]. \end{aligned}$$

The first best allocation based on verifiable types and returns in my model is the same as in Allen and Gale (1998), and solves the consumer's constrained maximization problem given for the Diamond and Dybvig (1983) framework in (4.1), with the substitution of

$$(1-\lambda)\, C_2(R) \leq \rho(1 - \alpha - \lambda C_1(R)) + \alpha R \tag{5.1}$$

for (4.1c). This substitution recognizes that when R is low, some goods available at $t = 1$ need to be consumed by late consumers for the optimal allocation.

The first-order conditions and binding constraints give the implicit first best solution $\widetilde{C}_1^*(R)$, $\widetilde{C}_2^*(R)$ and $\widetilde{\alpha}^*$ according to

$$
\begin{aligned}
E[u'(\widetilde{C}_1^*(R))] &= E[Ru'(\widetilde{C}_2^*(R))] && (5.2a)\\
\lambda \widetilde{C}_1^*(R) &= 1-\widetilde{\alpha}^*, \qquad (1-\lambda)\,\widetilde{C}_2^*(R) = \widetilde{\alpha}^* R \qquad \text{if } R \geq \overline{R} && (5.2b)\\
u'(\widetilde{C}_1^*(R)) &= \rho u'(\widetilde{C}_2^*(R)) \qquad \text{if } R < \overline{R}, && (5.2c)
\end{aligned}
$$

where

$$\overline{R} \equiv \frac{\rho\,(1-\alpha)\,(1-\lambda)}{\alpha\lambda}.$$

The first-order conditions are similar to that of Diamond and Dybvig (1983). (5.2a) is equivalent to (4.2a) but in expectation form since R is random. (5.2b) is the same as (4.2b) and (4.2c) for when returns are high, $R \geq \overline{R}$. When returns are low, $R < \overline{R}$, (5.2c) shows consumption must be shared between early and late consumers to equalize marginal utility (up to a factor of the storage rate of transformation ρ). This implies that $\widetilde{C}_2^*(R) = \rho\widetilde{C}_1^*(R)$. Since $\lambda > 0$, $\widetilde{\alpha}^* < 1$.

5.1. Banking Solution with Real-Goods Contracts

Allen and Gale (1998) show that banks cannot provide the first best solution with real-goods demand deposit contracts unless late consumers run the bank when R is observed to be low. Because contracts are payable in real goods, all stored goods must be paid out at $t = 1$, so goods cannot be shared with late consumers unless they withdraw early. When late consumers see at $t = 1$ that returns will be low at $t = 2$, there must be a partial run of the bank to achieve efficient allocation. The late consumers who run the bank ("runners") share in the $t = 1$ payout of stored goods with the early consumers in order to balance consumption per person between $t = 1$ and $t = 2$, as required by (5.2c).

When it is more efficient for the bank to store goods between $t = 1$ and $t = 2$ than for the runners to do so, or $\rho > 1$, a run would be inefficient. Demand deposit contracts are nominalized and the central bank must provide an injection of money through a loan to the bank. This deflates the contract because withdrawals at $t = 1$ are paid partly in goods and partly in money. Runners exchange their goods for money with the early consumers at a market clearing price. The bank is able to satisfy withdrawals while storing some goods until $t = 2$. The runners store money efficiently rather than store goods inefficiently. However, this still implies leakage from the banking system since runners hoard currency. The runners can then purchase the goods stored by the bank with their money at $t = 2$ at a market clearing price, and the bank repays the loan to the central bank.

5.2. Banking Solution with Nominal Contracts

I show that with nominal contracts, a unique equilibrium with first best results and no bank runs continues to hold under aggregate uncertainty of returns, for all $\rho \geq 1$.[18] The focus here is not on how the interbank market lends to prevent runs, as in the benchmark model. Rather, the key is that since deposits pay out nominal amounts, the bank can pay fixed promises in dollar terms, yet payouts are not inefficiently fixed in terms of real goods. Depositors' consumption can flexibly respond to aggregate shocks in the economy through prices, and the market can efficiently ration goods between early and late consumers due to the price mechanism. Late consumers do not have to run the bank when R is low to share in the relative abundance of goods at $t = 1$ because the entrepreneur stores goods over until $t = 2$.

I further modify my model to assume there is a single entrepreneur who is a price taker and combines the roles of the short term and long term entrepreneurs in the benchmark model. This assumption is necessary to ensure a zero profit condition for the entrepreneur. Without it, the long term entrepreneur suffers loses when $R < \overline{R}$, while the short term entrepreneur earns profits, so the long term entrepreneur could not expect to break even. The entrepreneur both stores and invests goods at $t = 0$, with debt repayments at $t = 0$ and $t = 1$. The bank can monitor the entrepreneur to ensure the proper amount of goods are stored at $t = 0$, since a critical function of the bank is to ensure optimal storage and investment.

For simplicity, I keep the same notation as with the benchmark model. Both of the superscripts "L" and "S" shall refer to the single entrepreneur and can be disregarded. For example, K_1^S refers to the entrepreneur's $t = 1$ debt repayment and K_2^L refers to his $t = 2$ debt repayment. If the entrepreneur defaults on K_1^S, the unpaid debt is carried over until $t = 2$.

At $t = 0$, the bank offers consumers the demand deposit contract (D_1, D_2) equal to (D, D), and lends one dollar to the entrepreneur for the debt contract (K_1^S, K_2^L), where

$$
\begin{aligned}
D_1 &= D_2 = D \equiv \widetilde{C}_1^*(\overline{R}) = \tfrac{1-\alpha}{\lambda} && (5.3)\\
K_1^S &= 1 - \alpha = \lambda D && (5.4)\\
K_2^L &= \tfrac{1}{\rho}\alpha\overline{R} = (1-\lambda)\, D && (5.5)\\
\alpha &= \widetilde{\alpha}^*.
\end{aligned}
$$

[18] The analysis could also be extended to aggregate uncertainty of liquidity, where λ is uncertain, to show that the first best outcome with no bank runs holds under nominal contracts similar to the results under aggregate uncertainty of returns. Diamond and Dybvig (1983) study aggregate liquidity shocks and claim that no bank contract with a sequential service constraint can give first best results under stochastic aggregate liquidity needs. In my model extended to aggregate liquidity shocks, nominal demand deposits would allow the bank to cover the entire aggregate liquidity shock by paying money for withdrawals, while an increase in the price level would provide optimal lower consumption to a larger number of early consumers. Since the aggregate shock is a real shock, efficiency requires that early consumers consume less. Thus, nominal contracts may uniquely allow bank demand deposits with a sequential service constraint to provide the first best outcome.

The last equalities in (5.3), (5.4) and (5.5) hold due to the first-order conditions in (5.2b). The bank requires that $1-\alpha$ of the loan must be used for storage of goods and α of the loan must be used for investment of goods, so $\beta_0^S = 1-\alpha$. What is required is to show that when $R \geq \overline{R}$,

$$\begin{aligned} P_1 &= 1 && \text{(5.6a)} \\ P_2 &= \frac{\overline{R}}{\rho R} && \text{(5.6b)} \\ \lambda^w &= \lambda^p = \lambda && \text{(5.6c)} \end{aligned}$$

is the unique equilibrium, so consumption is first best:

$$\begin{aligned} C_1(R) &= \frac{D}{P_1} = \frac{1-\alpha}{\lambda} = \widetilde{C}_1^*(R) && \text{(5.7a)} \\ C_2(R) &= \frac{D}{P_2} = \frac{(1-\alpha)\,\overline{R}}{\lambda\rho R} = \widetilde{C}_2^*(R). && \text{(5.7b)} \end{aligned}$$

This says that when $t = 2$ returns are high, late consumers optimally consume more, reflected by low $t = 2$ prices. Also required to show is that when $R < \overline{R}$,

$$\begin{aligned} P_1 &= \frac{\rho\,(1-\alpha)}{\lambda[(1-\alpha)\,\rho + \alpha R]} && \text{(5.8a)} \\ P_2 &= \frac{1-\alpha}{\lambda[(1-\alpha)\,\rho + \alpha R]} && \text{(5.8b)} \\ \lambda^w &= \lambda^p = \lambda && \text{(5.8c)} \end{aligned}$$

is the unique equilibrium, so consumption is first best:

$$\begin{aligned} C_1(R) &= \frac{D}{P_1} = (1-\alpha) + \frac{1}{\rho}\alpha R = \widetilde{C}_1^*(R) && \text{(5.9a)} \\ C_2(R) &= \frac{D}{P_2} = \rho\,(1-\alpha) + \alpha R = \widetilde{C}_2^*(R). && \text{(5.9b)} \end{aligned}$$

This says that when $t = 2$ returns will be low the market will store goods over from $t = 1$ to $t = 2$ in order to equalize prices, giving early and late consumers equal consumption (to a factor of ρ).

The first best outcome depends on the entrepreneur choosing to store β_1^L goods and sell $1-\alpha-\beta_1^L$ goods at $t = 1$, and choosing to sell all of his remaining goods at $t = 2$. This requires

$$\beta_1^L = 0 \qquad \text{if } R \geq \overline{R} \tag{5.10}$$

$$\beta_1^L = (1-\lambda)(1-\alpha) - \frac{1}{\rho}\lambda\alpha R \qquad \text{if } R < \overline{R} \tag{5.11}$$

$$Q_2^L = \alpha R + \beta_1^L \rho. \tag{5.12}$$

The return on the interbank loan is always one, or $D_{1,2}^{ff} = 1$, since $D_1 = D_2$. This implies the demand deposit at the second bank also pays one, $D_{1,2} = 1$. This is shown below in Lemma 3.

The entrepreneur's optimization problem at $t = 1$ is to maximize his total consumption of goods. This involves a combination of constraints from the short term and long term entrepreneurs' optimizations in the benchmark model:

$$\max_{\beta_1^L, Q_2^L} \quad [\overline{Q}_2^L(\beta_1^L) - Q_2^L \mid \lambda^w, \lambda^p] \tag{5.13a}$$

$$\text{s.t.} \quad \text{(4.11b), (4.10b), (4.10c) and}$$

$$\beta_1^L \leq \beta_0^S \tag{5.13b}$$

with the requirement that β_1^L and Q_2^L are nonnegative. A single entrepreneur changes the definition of some variables. Redefine $\overline{Q}_2^L \equiv \alpha R + \beta_1^L \rho$, $Q_1^S \equiv \beta_1^S - \beta_1^L$, and $M_2^L \equiv K_1^S + K_2^L - Q_1^S P_1 D_{1,2}$. S_2^S, Q_1^L and β_1^S are not used.

Prices are as defined in the benchmark model, but can be expressed more simply as:

$$P_1 = \frac{\lambda^p D}{Q_1^S}$$

$$P_2 = \frac{(1-\lambda^p) D}{Q_2^L}.$$

If the bank is not expected to default, late consumers do not withdraw and redeposit at the second bank at $t = 1$ because the second bank can never pay more than the original bank: $D_1 D_{1,2} = D_2$. If $P_2 \leq P_1$, late consumers choose to purchase goods at $t = 2$, so $\lambda^p = \lambda$.[19] If $P_1 < P_2$, late consumers choose to withdraw and purchase goods early at $t = 1$, so there is a run and $\lambda^w = \lambda^p = 1$. Intermediate cases of $\lambda < \lambda^p < 1$ require that $P_1 = P_2$. If at $t = 1$,

[19] As in the benchmark model, if late consumers were to withdraw early and redeposit or buy goods when they are indifferent between that or not withdrawing, these inconsequential withdrawals could exist in equilbrium but they would not be properly described as "runs" (since in Allen and Gale (1998) runs mean that late consumers strictly prefer to withdraw early and must do so for a first best outcome). Redeposited money would simply be lent back to the original bank. Late consumers buying goods at $t = 1$ when $R < \overline{R}$ implies the entrepreneur would not store those goods over to $t = 2$.

any or all late consumers expect the bank to default, there may be more late consumers who withdraw at $t = 1$ than when late consumers do not expect the bank to default. This implies that λ^w and λ^p may be greater but not less for an expected bank default than for no expected bank default, given $\frac{P_2}{P_1}$.

The following lemma is identical to Lemma 2 in showing that the bank never defaults for all λ^w and λ^p due to efficient interbank lending.

Lemma 3. *The original bank always receives any needed loan L_1^B from the second bank and never defaults on the loan repayment or depositor withdrawals at $t = 1$ or $t = 2$, for all λ^w and λ^p. The return on the loan and on deposits made at $t = 1$ are $D_{1,2}^{ff} = D_{1,2} = 1$.*

Proof. See Appendix. ■

The final step is to solve for the entrepreneurs' choices over the quantity of goods to sell each period, incorporating the late consumers' actions as a function of prices. This determines market prices and consumer actions.

Suppose all late consumers withdraw and purchase goods at $t = 2$. Consider the case of $R \geq \overline{R}$. This implies $P_2 \leq P_1$. If the entrepreneur were to store goods at $t = 1$ in order to sell less at $t = 1$ and more at $t = 2$, P_1 would increase and P_2 would decrease. Thus, the entrepreneur never chooses to store goods at $t = 1$. The optimal allocation does in fact call for late consumers to receive greater consumption, which occurs since additional goods are produced at $t = 2$ due to the high return on investment. Thus, late consumers do not run the bank.

Now consider the case of $R < \overline{R}$. The entrepreneur does store goods at $t = 1$ to sell at $t = 2$, and $P_1 = \rho P_2$. At these prices, the entrepreneur stores just enough goods so that the marginal rate of transformation, ρ, equals the real price of $t = 2$ goods in terms of $t = 1$ goods, $\frac{P_1}{P_2} = \rho$. Thus, since investment returns are low, the goods market supplies enough additional goods at $t = 2$ so that late consumers do not run the bank to buy goods at $t = 1$. Thus, no runs occur.

Proposition 2. *The unique equilibrium of the benchmark model extended to aggregate uncertainty of returns is the first best outcome $C_1(R) = \widetilde{C}_1^*(R)$, $C_2(R) = \widetilde{C}_2^*(R)$ and $\alpha = \widetilde{\alpha}^*$, with no bank runs.*

Proof. First, I solve the entrepreneur's optimizations to show (5.10) and (5.11) is the unique solution. (See Appendix).

Second, I show that at $t = 2$, the entrepreneur sells all of his goods, $Q_2^L = \overline{Q}_2^L$. Suppose not, $Q_2^L \neq \overline{Q}_2^L$. This implies $S_2 = M_2^L = 0$ from Lemma 3.1, which is given in the appendix.

But

$$M_2^L = K_2^L + K_1^S - Q_1^S P_1 = D - \lambda^p D = 0$$

implies $\lambda^p = 1$, a contradiction to $\lambda^p = \lambda < 1$. Thus, $Q_2^L = \overline{Q}_2^L$.

Third, I show that all late consumers withdraw and purchase goods at $t = 2$, $\lambda^w = \lambda^p = \lambda$. By Lemma 3, the bank never defaults, so by the late consumers' optimization, they withdraw at $t = 2$, so $\lambda^p = \lambda^w$.

For $\lambda^p = \lambda$, if $R \geq \overline{R}$, $\beta_1^L = 0$ implies $P_1 = \frac{\lambda D}{\lambda D} = 1$ and $P_2 = \frac{(1-\lambda)D}{\alpha R} = \frac{\overline{R}}{\rho R} \leq 1$. Thus $P_1 \geq P_2$, so $\lambda^p = \lambda$ by the late consumers' optimization. If $R < \overline{R}$, $\beta_1^L = (1-\lambda)(1-\alpha) - \frac{1}{\rho}\lambda\alpha R$ implies

$$\begin{aligned} P_1 &= \frac{\lambda D}{1-\alpha-\beta_1^L} = \frac{1-\alpha}{\lambda(1-\alpha+\frac{1}{\rho}\alpha R)} \\ P_2 &= \frac{(1-\lambda)D}{\alpha R + \rho\beta_1^L} = \frac{1-\alpha}{\lambda[\rho(1-\alpha)+\alpha R]}. \end{aligned}$$

Thus, $P_1 = \rho P_2$, so $P_1 \geq P_2$, and $\lambda^p = \lambda$ by the late consumers' optimization. Thus, $\lambda^w = \lambda^p = \lambda$.

Finally, I show that the equilibrium is unique and the outcome is first best. Since (5.10), (5.11), (5.12) and $\lambda^w = \lambda^p = \lambda$ are uniquely determined, if $R \geq \overline{R}$, (5.6a) and (5.6b) are uniquely determined, so (5.7) is also; and if $R < \overline{R}$, (5.8a) and (5.8b) is uniquely determined, so (5.9) is as well. Thus, the outcome is first best, and the consumer's incentive constraint, $C_2 \geq C_1$, is satisfied. This confirms the conjecture of the bank's choice of contracts offered. Thus, $C_1(R) = \widetilde{C}_1^*(R)$, $C_2(R) = \widetilde{C}_2^*(R)$ and $\alpha = \widetilde{\alpha}^*$ is the unique equilibrium. ■

6. No Banks Runs with Currency

I introduce currency to the benchmark model to allow late consumers to withdraw currency at $t = 1$ and hoard it until $t = 2$. I show that allowing for leakage due to currency hoarding alone does not imply that Diamond-Dybvig style runs occur. Rather, this approach identifies circumstances that do cause these runs. Bank runs would also not occur if currency withdrawals were allowed in the model extended to aggregate uncertainty of returns. Since the bank never defaults and late consumers receive the first best consumption without running the bank, allowing them to withdraw currency early would change nothing.

The significance of the argument in this section is that bank runs and failures in a modern economy due to currency withdrawals may not be the most important threat. The most alarming bank failure in recent times (and largest ever) in the U.S. was Continental Illinois in 1984, which failed due to a run by large depositors withdrawing using wire transfers. Actually, the bank had no retail branches so currency outflows were not an issue. In general, though, if large depositors are concerned with a potential bank failure, withdrawing currency is too impractical. Rather, large wholesalers use wire transfers to withdraw. Moreover, large currency withdrawals require more time to fulfill than wire transfers so there would be much more risk that the bank would fail before the withdrawal is completed. Bank failures due to currency runs are more descriptive of 19th and early 20th century bank panics, as described by Gorton (1988). My no-bank runs results would not apply to this era since same-day clearing of electronic payments was not available then. It may be that deposit insurance in the U.S. since the Great Depression has precluded currency runs which would otherwise have taken place. However, many bank-type institutions have not been insured or have had less than fully trusted state-insurance. Runs at state-insured thrifts in Ohio and Maryland in the mid-1980s are noted as the first bank runs since the Great Depression in which customers were lining up for physical withdrawals (Wolfson, 1994), demonstrating that currency runs have not been typical of bank runs in the era of modern payment systems. Moreover, deposit insurance does not protect large depositors who are beyond the coverage limit, and withdrawals by these depositors are often decisive for a bank collapse.

I use a simple extension to the benchmark model to allow for consumers and banks to hold currency and make payments with currency as well as with electronic payments. The bank stores $1 - \alpha$ in currency that is deposited by consumers who receive it from the central bank at $t = 0$ rather than lend it to the short term entrepreneur. The currency held by the bank may be considered as partial reserves held due to the bank's uncertain liabilities at $t = 1$, which depend on the fraction of early withdrawals λ^w. Reserves also may be held due to the bank's illiquid investments of loans to the long term entrepreneur that are not repaid until $t = 2$. Reserves may be considered either mandated by the central bank or voluntarily held. At $t = 1$, the bank first pays currency to those who withdraw. If there is no run, withdrawals of λD_1 equal the currency held. After $t = 1$ withdrawals, the bank no longer holds currency as reserves according to the interpretation above, since the uncertainty over withdrawals is resolved and investments are liquid since they will be repaid the following period ($t = 2$).

Since the central bank receives $1 - \alpha$ in goods from consumers at $t = 0$ but does not sell the goods to the short term entrepreneur as in the benchmark model, it must play an expanded role in the model in order for these goods to be returned to the market. The method I use is a simple construction for the goods to be sold in the market and the outstanding currency to be redeemed, even though after $t = 0$ the central bank has dropped its "gold standard" guarantee to exchange goods at a fixed price. I assume the central bank sells its goods at the market price at $t = 1$ or $t = 2$. Since currency is an outstanding liability of the central bank, but its value is no longer fixed, I assume the central bank will accept currency for goods, but only at the current market price.

Specifically, the central bank acts as a competitive price taker and profit maximizer (in goods consumed at $t = 2$) subject to the obligation of redeeming currency for goods at market prices at $t = 1$ and $t = 2$ when possible. At $t = 1$, the central bank sells its goods at the market price for currency. The central bank will also accept electronic money—electronic payments of funds—to buy goods at $t = 2$ if profitable. The central bank keeps an account at the second bank to hold any electronic funds it receives at a return of $D_{1,2}$ on deposits made at $t = 1$.[20] At $t = 1$ or $t = 2$, the central bank will also exchange any electronic money it has for currency. Any currency the central bank receives will be retired and not re-spent. Finally, if the central bank holds excess electronic funds at $t = 2$ that are not spent to redeem outstanding currency, it will spend all of these funds to purchase and consume goods as a competitive buyer.

The central bank accepts currency as long as it has goods or electronic funds to exchange for currency, so the maximum net amount of money the central bank receives over $t = 1$ and $t = 2$ is the amount of currency issued, $1 - \alpha$. Thus, the total demand for money by the central bank is the same as that by the short term entrepreneur in the benchmark model without currency. Hence, currency can be included in the model and hold value due to the credit/debit payment value of money even though its value is not guaranteed by the central bank. Analytically, the model is the same as the benchmark model without currency, with the exception that the short term entrepreneur is replaced by the central bank. The optimization problem is the same except that the short term entrepreneur's loan repayment K_1^S due to the bank is replaced with the central bank's equally sized liability due to itself for retiring currency at $t = 1$ or $t = 2$.

I examine currency withdrawal from the bank under two different sets of assumptions:

A1 The bank pays withdrawals with currency until it is depleted and then the bank pays with electronic payments.

A2 The bank has to pay withdrawals with currency if it is demanded, or else the bank defaults and must call its loans, causing full liquidation, and then pays the remaining withdrawals with electronic payments. Additionally, consumers believe that no one takes weakly dominated actions.

Consider assumption (A1). If there were a run, $\lambda^w > \lambda$, the bank pays the first λD_1 of withdrawals in currency and the remaining $\lambda^w D_1 - \lambda D_1$ of withdrawals in electronic payments. Suppose late consumers believe at $t = 1$ that other late consumers will withdraw and hoard currency at $t = 1$. If the bank can pay the remaining withdrawals with electronic payments, it

[20] If the central bank instead had an electronic funds account at its own individual "bank" with the policy to lend all net positive funds at the market interbank lending at $t = 1$, all results would be the same.

can receive any needed loans from the second bank and it does not default. Thus, the marginal late consumer would prefer to withdraw at $t = 2$. Hence, no late consumers prefer to withdraw early and there is no bank run.

Next, consider assumption (A2), that the bank must pay currency if demanded or else default. I examine a marginal late consumer's strategy given the actions of other late consumers. I show that when there exists a safe bank to redeposit with, demanding and hoarding currency is a weakly dominated strategy. Late consumers are indifferent to buying goods later with currency or immediately with electronic payments, and they prefer redepositing currency with the second bank over hoarding it. By (A2), this implies consumers expect that no one demands currency, which implies that the bank would not fail due to excess currency demands. Even if late consumers were still to run the bank and redeposit electronic withdrawals at the second bank, the original bank can borrow and does not fail. Thus, late consumers do not prefer to withdraw early and runs do not materialize. Note that this result does not depend on any assumption that the currency that is withdrawn and redeposited can be lent back to the bank within the same period. I assume currency transactions take too much time for this to occur. Only electronic redeposits can be relent the same day. The need to impose restrictions on beliefs in assumption (A2) makes the result of no runs when banks must pay currency on demand not as strong, so I do not claim currency runs can never occur under efficient interbank lending. Rather, the goal is to delineate the circumstances required for Diamond-Dybvig runs to occur and argue that they are not the main threat in a modern banking system. A condition under which currency-demand runs could occur is if this assumption on beliefs does not hold. In this case, running the bank and hoarding currency is a weakly dominated action when other late consumers withdraw early. Regardless, this is a contrast from Diamond and Dybvig (1983), in which running the bank and hoarding goods are a strictly preferred action when other late consumers withdraw early.

There are several conditions that together would allow for a Diamond-Dybvig depositor run to occur, even given the assumption in (A2) that consumers believe no one takes weakly dominated actions. The first condition is that all banks (i.e. the original bank and the second bank) hold deposits and illiquid loans from $t = 0$. This implies that it must not be possible for any new bank to be created during the panic when a new bank would be desired, otherwise this bank would provide consumers a safe place to redeposit currency since it could not be run. The bank would then act as a coordination device to resolve the potential run. All late consumers would redeposit with the safe bank and not demand currency from their own banks. Thus, the banks do not fail from currency runs and so the runs would unwind. The second condition is that late consumers believe that all of the banks are being run. If there were a single bank not expected to be run, consumers would redeposit there and again bank runs would in turn never occur. The third condition is that banks are required to pay currency on demand rather than be able to pay electronic payments (or "bank checks"), assumption (A2), and that late consumers must demand currency for withdrawal, which causes banks to default.

If a panic were to occur according to these conditions, the central bank could resolve it by acting as lender of last resort to guarantee just a single bank, which would coordinate taking redeposits and then relending so a run would not occur. This allows for a different policy than deposit insurance or suspension of convertibility as proposed by Diamond and Dybvig (1983) as a solution to depositor runs. It is important that the lender of last resort role can resolve depositor runs, because a lender of last resort is also shown in the next section to resolve banking crises at the interbank level.

Proposition 3. *Under assumption* (A1) *or* (A2)*, the unique equilibrium of the benchmark model with nominal contracts extended to include currency withdrawals is the same first best outcome with no bank runs as in the benchmark model without currency.*

Proof. For assumption (A1), see the Appendix. Under assumption (A2), suppose all consumers were to withdraw at $t = 1$, demand currency, and not redeposit with the second bank. The bank would default since it would not have the currency to meet the demand. Let δ_1 be the fraction paid of the demand deposit amount owed to consumers who withdraw from the bank at $t = 1$, where $\delta_1 \leq 1$. Under a pro-rata rule, $\delta_1 D_1$ is paid to each consumer who withdraws at $t = 1$. Under a sequential-service constraint, δ_1 is the fraction of consumers attempting to withdraw at $t = 1$ that receive D_1, and all others receive nothing. Since the bank defaults, it must recall the long term loan K_2^L from the long term entrepreneur, so $\gamma_1^L = \alpha$. For simplicity, assume consumers have the typical coefficient of relative risk aversion (CRRA) greater than one.

Once the bank has defaulted and has paid out all currency, consumers accept electronic payments to purchase goods at $t = 1$ as well. The bank liquidates all loans, so the long term entrepreneur sells αr goods at $t = 1$ and pays all proceeds to the bank. The bank pays λD_1 in currency and $P_1 \alpha r$ in electronic money. Prices are given by

$$\begin{aligned} P_1 &= \frac{\delta_1 \lambda^p D_1}{\alpha r + Q_1^S} \\ P_2 &= \frac{\delta_1 (1 - \lambda^p) D_1}{Q_2^S}, \end{aligned}$$

where Q_t^S is the quantity of goods sold by the central bank at time t. Based on the consumer's and central bank's optimizations, $P_1 = P_2$. If $P_1 > P_2$, late consumers would purchase goods from the central bank with currency at $t = 2$ only, driving P_2 up. If $P_2 < P_2$, the opposite would occur. The electronic money that the long term entrepreneur repays to the bank is the amount of money the bank can pay to consumers and so must equal the electronic money consumers can pay for goods at $t = 1$. In addition, the amount of currency paid by consumers to the central bank equals the amount of goods the central bank has available to sell. Thus, prices equal one.

Late consumers buy goods from the central bank with their currency, so they are indifferent between buying these goods at $t = 1$ and $t = 2$. All consumers have equal consumption since they all withdraw $\delta_1 D_1$, so consumption is equal to the total goods available divided by the number of consumers (which is normalized to one), thus $C_1 = 1 - \alpha + \alpha r < 1$. A marginal late consumer is indifferent between i) buying goods at $t = 1$, ii) hoarding his currency and buying goods at $t = 2$, and iii) redepositing his currency with the second bank and buying goods at time $t = 2$, since the second bank never defaults. Thus, he is also indifferent between demanding currency to make the original bank default at $t = 1$, and accepting only an electronic payment for goods at $t = 1$.

Suppose instead all consumers were to withdraw at $t = 1$ but not demand currency, and late consumers were to redeposit the fraction $\sigma \in [0, 1]$ of the currency they receive with the second bank and to purchase goods at $t = 1$ with any electronic funds they receive. The bank would not default since there is no excess demand for currency, and it would be able to borrow all of the electronic funds it needs from the second bank. Prices are given by

$$\begin{aligned} P_1 &= \frac{\lambda D_1 + (1 - \lambda) D_2 - \sigma \tau \lambda D_1}{Q_1^S + Q_1^L} \\ P_2 &= \frac{\sigma \tau \lambda D_1 D_{1,2}}{Q_2^L}, \end{aligned}$$

where $\tau \in [0, 1]$ is the fraction of total currency that is paid to late consumers. A marginal late consumer prefers to redeposit all of the currency he receives at the second bank since he receives interest $D_{1,2} \geq 1$ and the second bank never defaults. He also prefers not to demand currency to force the original bank to default.

Finally, suppose all late consumers withdraw at $t = 2$ and do not demand currency. The outcome is first best, so a marginal late consumer prefers not to demand currency at $t = 1$ or $t = 2$, but rather to buy goods at $t = 2$ with an electronic payment.

Since the marginal late consumer either is indifferent to or prefers not to demand currency and force the bank to default at $t = 1$ or $t = 2$, given any actions by other consumers, demanding currency is a weakly dominated action. Thus, if late consumers believe according to (A2) that no consumers take weakly dominated actions, then they believe that the bank never defaults. Given this, the marginal late consumer prefers to withdraw at $t = 2$ for all λ^w and λ^p. Thus, all late consumers prefer to withdraw at $t = 2$, there are no bank runs, and the equilibrium is the first best outcome. ■

7. Inefficient Interbank Lending and Banking Crises

7.1. Banking Crisis Model

Consider now the benchmark model extended to include multiple banks that issue consumer deposit accounts and entrepreneur loans at $t = 0$. If there are local liquidity shocks to individual banks, a bank may need to borrow from more than one other bank. Without the assumption of efficient interbank lending, a lending coordination issue may arise. Either all banks lend to the bank in need for a first best outcome, or no banks lend, which leads to inefficient liquidation, price deflation and a possible bank run. I show that the liquidity of all banks is reduced and all consumers have inefficient consumption sharing. A large enough shock may cause contagion and runs at all banks.

The benchmark model with no aggregate shock and no currency is expanded to include three banks that have deposits and loans at $t = 0$. Denote the banks $i \in \{A, B, C\}$. Each bank takes deposits from consumers at $t = 0$ and has a fraction of early consumers λ^i. Consumers at each bank have consumption of either C_1^i or C_2^i. Bank i issues a loan to a single price-taking entrepreneur i. An entrepreneur with both short term and long term investments is necessary to ensure a zero profit condition for the entrepreneur. Denote all variables previously denoted with the superscript "S" or "L" for short term or long term entrepreneur instead with the superscript "i", where $i \in \{A, B, C\}$, to refer to the entrepreneur with loan from bank i. Loan repayments of K_1^i and K_2^i are the same, with the addition of a callable-loan rule that allows the bank to recall the amount of K_2^i necessary at $t = 1$ to prevent default. Each bank can enforce the fraction of the loan its entrepreneur uses to buy and store goods.

While the economy is triple in the size of consumers, goods and entrepreneurs, the first best results are the same as in the benchmark model. Each bank offers the first best demand deposit contract (D_1, D_2) to its consumers. The bank also contracts with its depositors at $t = 0$ that demand deposit contracts $D_{1,2}^i$ offered at $t = 1$ pay a return less than or equal to $\frac{D_2}{D_1}$. This ensures that $t = 1$ depositors do no expropriate $t = 0$ depositors. The first best also requires that each bank chooses $\beta_0^i = 1 - \alpha^*$ of its loan to be stored by the entrepreneur in goods. However, Skeie (2003a) shows that under nominal contracts with multiple banks, banks do not hold the optimal amount of liquid short term loans[21]. To resolve the underprovision of

[21] Skeie (2003) shows that under nominal contracts, banks do not hold optimal liquidity, while under real contracts banks do. The previous real-contracts literature has shown that banks do not hold optimal liquidity, but this is only due to additional frictions. (See Bhattacharya and Gale, 1987; Bhattacharya and Fulghieri, 1994; and Holmström and Tirole, 1998). With nominal contracts and no additional frictions, banks do not invest in the first best amount of storage because they can free-ride off of other banks' storage investment. They can do this since they do not owe a fixed amount of real goods, but rather due to nominal contracts they owe a fixed payment of money which can be borrowed in the interbank market. While nominal contracting shows how bank runs that would occur under real contracts are resolved, it shows how the underprovision of liquidity, which

liquidity, assume each bank is required by the government to choose $\beta_0^i = 1-\alpha^*$ of the loan to be stored in goods. For simplicity, assume again that consumers have a coefficient of relative risk aversion (CRRA) greater than one.

Local liquidity shocks occur due to the uncertain fraction of early consumers λ^i for each bank. Specifically,

$$\begin{aligned} \lambda^A &= \lambda + 2\epsilon \\ \lambda^B &= \lambda^C = \lambda - \epsilon, \end{aligned}$$

where ϵ is a random variable with c.d.f. $F(\cdot)$ and $E[\epsilon] = 0$, so each bank has no expected shock, $E[\lambda^i] = \lambda$. To ensure $\lambda^i \in (0,1)$, ϵ has support $\left[\max\left\{-\frac{\lambda}{2}, \lambda - 1\right\}, \min\left\{\lambda, \frac{1-\lambda}{2}\right\}\right]$. However, there is no aggregate shock. The total fraction of early consumers is

$$\lambda^A + \lambda^B + \lambda^C = 3\lambda.$$

If $\epsilon > 0$, bank A needs to borrow from both banks B and C. If $\epsilon < 0$, bank A is able to lend to both banks B and C. The assumption of efficient interbank lending is relaxed. To be concrete, the lending game is modeled as follows. At $t = 1$, bank $i \in \{B, C\}$ privately offers bank A a loan of L_1^i at a return of $D_{1,2}^{ff,i}$. ($L_1^i < 0$ implies bank i asks to borrow from bank A). Bank A observes both offers and then accepts or rejects each. The actual lending is then publicly revealed.

Let $\lambda^{w,i} \geq \lambda^i$ be the fraction of consumers who withdraw from bank i at $t = 1$. Let δ_1^i be the fraction paid of the demand deposit amount owed to consumers who withdraw from bank i at $t = 1$, where $\delta_1^i \leq 1$. For simplicity, assume a pro-rata rule, so $\delta_1^i D_1$ is paid to each consumer who withdraws at $t = 1$. From the original assumptions, if $\delta_1^i < 1$, bank i defaults at $t = 1$ and must recall the long term loan K_2^i from entrepreneur i, so $\gamma_1^i = \alpha$, where γ_1^i is the amount of entrepreneur i's invested goods at $t = 0$ that are liquidated at $t = 1$. Let $\delta_2^i D_2$ be the money paid to a consumer who withdraws at $t = 2$ from bank i, where $\delta_2^i \leq 1$. If $\delta_2^i < 1$, then bank i defaults at $t = 2$.

Prices are given by

$$\begin{aligned} P_1 &\equiv \frac{\left(\delta_1^A \lambda^{p,A} + \lambda^{p,B} + \lambda^{p,C}\right) D_1}{Q_1^A + Q_1^B + Q_1^C} \\ P_2 &\equiv \frac{S_2}{Q_2^A + Q_2^B + Q_2^C}, \end{aligned}$$

real contracts rule out, is pervasive. The underprovision of liquidity can be resolved by government mandated reserve requirements (which can be held in currency as in the currency extension in this paper) or mandated loan portfolio balance.

where S_2 is the total money spent on goods by late consumers at $t = 2$, and $Q_1^i \equiv \beta_0^i - \beta_1^i + \gamma_1^i r$ is redefined to correspond to the assumption of a combined short and long term entrepreneur.

For any value of ϵ, the aggregate fraction of early consumers is constant. Under efficient lending among banks A, B and C, first best results obtain identically as in the benchmark model. When $\epsilon > 0$, bank A needs to borrow from banks B and C and multiple equilibria arise. Either both banks B and C lend to A, or neither lend. If bank A cannot borrow funds from banks B and C, it must recall at least part of the long term loan in order to pay depositors who withdraw at $t = 1$, forcing entrepreneur A to liquidate goods. Due to a single entrepreneur, when $P_1 < 1$, the entrepreneur would have to repay K_1^i by inefficiently liquidating assets. When banks B and C do not lend to bank A, I show below that $P_1 < 1$ for $\epsilon > 0$. Entrepreneurs for all banks would fully liquidate investments, and all banks would have complete bank runs. To avoid this, I allow for modest renegotiation of K_1^i by assuming $(K_1^i - P_1 \lambda D_1)^+$ can be repaid at $t = 2$ at a return of $\frac{D_2}{D_1}$ as long as the bank does not default.

When $\epsilon > 0$, if bank A cannot borrow from banks B and C, some liquidation by entrepreneur A must occur. This causes deflation and P_1 falls. There are fewer goods on the market at $t = 2$, so P_2 rises. The drop in P_1 and increase in P_2 implies that consumers from all banks have inefficient ex-ante risk sharing. Late consumers from all banks consume less than the first best. Early consumers from all banks consume more than the first best for small shocks and less than the first best for large enough shocks.

As stated above, the drop in P_1 implies that entrepreneurs $i \in \{B, C\}$ cannot pay their loans K_1^i in full at $t = 1$. Whereas in the lending equilibrium banks B and C receive excess balances at $t = 1$ that they then lend out, in the no-lend equilibrium banks B and C are repaid less from entrepreneurs on $t = 1$ loan repayments so banks B and C not receive excess balances. When coordinated lending by banks B and C breaks down, they actually lose the excess reserves to lend. In effect, when bank A cannot borrow the funds it needs from banks B and C, it competes to capture a larger share of funds available in the goods market by liquidating goods.

These results of the model illustrate the nature of money and lending as an endogenous liquidity flow. When some banks stop lending, liquidity dries up in the sense that other banks lose their ability to lend. For instance, when bank B lends to bank A, out of the loan bank A makes payments to entrepreneurs for bank A's consumer purchases. Some of the payments go to entrepreneurs B and C and are deposited at banks B and C. This allows bank C as well as bank B to in turn lend to bank A. Hence, there are positive externalities accruing to all banks from bank B lending to bank A. In the U.S., \$154 billion is lent for an overnight term in the interbank market every day from base Federal Reserve account bank balances of only \$15 billion. This means that many banks can lend only because other banks have previously lent or made payments to them during the day. Once some banks stop lending, other banks cannot borrow and relend or make payments as normal, so the flow of money can be greatly reduced.

Proposition 4. *If bank A needs liquidity from multiple banks ($\epsilon > 0$), then there exists multiple equilibria:*
i) Banks B and C both lend to bank A and the first best outcome obtains: $C_1^i = C_1^*$, $C_2^i = C_2^*$ and there are no bank runs, corresponding to the benchmark model.
ii) Neither bank lends to bank A, bank A liquidates invested goods at $t = 1$ and defaults at $t = 2$, and consumption sharing for consumers of all banks is suboptimal.

Proof. See appendix. ■

When bank A recalls some of its loans, entrepreneur A must liquidate invested goods to repay at $t = 1$. This implies that bank A receives a lower repayment at $t = 2$ and cannot fully repay depositors at $t = 2$, so $\delta_2^A < 1$. However, if the ϵ shock is small enough that bank A does not have to recall a very large amount of the long-term loan, then bank A late consumers are better off not running the bank at $t = 1$ because they still receive more by waiting until $t = 2$. They only run the bank and purchase goods at $t = 1$ if $\frac{D_1}{P_1} > \frac{\delta_2^A D_2}{P_2}$. Bank A late consumers would not prefer to withdraw and redeposit at bank B or C when bank A will default at $t = 2$. Since banks B and C are not lending to bank A, they would not receive any interest on funds redeposited from bank A's late consumers, so they do not offer positive interest rates on redeposits from bank A's late consumers. In other words, the return $D_{1,2}^i$ promised by any bank on money deposited at $t = 1$ equals one.

The next proposition shows that in the no-lend equilibrium, when the shock to bank A is large enough such that $\frac{D_1}{P_1} > \frac{\delta_2^A D_2}{P_2}$, late consumers run bank A and it must recall its entire loan, forcing full liquidation. Bank A is not able to repay consumers fully at $t = 1$ so $\delta_1^A < 1$.

Proposition 5. *If*

$$\epsilon > \widetilde{\epsilon}^A \equiv \frac{r\lambda (1-\lambda)(D_2 - D_1)}{\lambda(3R - 2r)D_1 + r(1-\lambda) D_2}$$

and banks B and C do not lend to bank A, then there is a complete bank run of bank A, and bank A recalls its entire loan and defaults at $t = 1$.

Proof. See appendix. ■

The decrease in $\frac{P_1}{P_2}$ discussed above occurs regardless of whether bank A has a run. When there is no run, there is a greater amount of goods on the market with no additional dollars to pay for goods at $t = 1$, forcing P_1 down. If bank A is run, the bank defaults at $t = 1$ and does not pay consumers in full, so $\delta_1^A < 1$. The additional supply of goods flooded onto the market from liquidation is greater than the additional supply of money paid to withdrawing consumers, so P_1 still falls. In either case, at $t = 2$, costly liquidation implies the reduction in goods on the market is less than the reduction in money supplied by late consumers, so P_2 rises.

When bank A is run, if P_1 is low enough then some late consumers of banks B and C may run their banks. This occurs because late consumers of banks B and C will run if $\frac{D_1}{P_1} > \frac{D_2}{P_2}$. These partial runs show how the illiquidity of bank A and its loan liquidation causes illiquidity in the market and contagion of bank runs to other banks.

Proposition 6. *If*

$$\epsilon > \widetilde{\epsilon}^{B,C} \equiv \frac{\lambda (1-\lambda)(D_2 - D_1)}{\lambda D_1 + (1-\lambda) D_2}$$

and banks B and C do not lend to bank A, then there is a complete bank run of bank A and there are partial bank runs of banks B and C.[22]

Proof. See appendix. ■

These results may shed light on the spiraling effect of bank troubles and price deflation, such as in Japan. When an initial individual bank shock leads the bank to liquidate loans, a less liquid market creates deflation and causes other banks' loans to lose value due to their own entrepreneurs defaulting. Thus, illiquidity on the liability side of banks can spill over to illiquidity and loss of value on the asset side of other banks, which may cause illiquidity on the depositor side of these other banks as their depositors withdraw. Initial banks that fail due to illiquidity can lead to other banks failing due to a feedback loop between the liability and asset sides of the banking system.

An important assumption is that entrepreneur A cannot borrow at $t = 1$ from banks B or C in order to pay bank A. The justification is that bank A has built a lending relationship with entrepreneur A so banks B and C could not collect on a loan made to entrepreneur A at $t = 1$. If entrepreneurs could always borrow with no restrictions from other banks when their loans are recalled, there is never a banking crisis. Since entrepreneur A represents many small entrepreneurs, each could borrow the entire amount needed at $t = 1$ from either bank B or C, so there would be no coordination problem between banks B and C to lend to entrepreneurs. To the extent that entrepreneurs can borrow partially from other banks, the degree of the banking crisis may be reduced.

[22] Since $\widetilde{\epsilon}^{B,C} > \widetilde{\epsilon}^{A}$, banks B and C are run only if bank A is run.

7.2. Central Bank Intervention

The multiple equilibria problem in the banking crisis model is analogous to that in Diamond and Dybvig (1983), but the coordination failure happens at the interbank level rather than at the depositor level. In Diamond and Dybvig (1983), late consumers either all keep deposits at the bank or all withdraw early. In the interbank problem, banks either all lend to the bank in need or all refuse to lend.

However, government solutions for depositor-based bank runs, such as deposit insurance and suspension of convertibility, do not solve interbank-based banking crises. In the banking crisis model, the central bank can resolve the banking crisis by guaranteeing to be lender of last resort. A lender of last resort is also a solution to depositor-based runs, as shown in Section 6 with currency withdrawals above. The role of lender of last resort for depositor runs is to coordinate late consumer redepositing to a bank that is backed by the lender of last resort. The role of lender of last resort in an interbank market crisis is to coordinate bank lending. When there is a lender of last resort to bank A, either bank B or C can lend to bank A even if the other does not. This is because they each know the lender of last resort will lend to bank A if it is still in need. Bank A will not default and the lending bank is assured to be repaid. In fact, banks B and C strictly prefer to lend to bank A. Due to the lender of last resort, bank A will not recall loans and liquidate goods. Thus P_1 does not fall and banks $i \in \{B, C\}$ are repaid on loans K_1^i at $t = 1$ fully. Then, banks B and C have a liquidity shock in effect at $t = 2$ since they have a larger number of late consumers withdrawing, $1 - \lambda + \epsilon$. If bank B or C does not lend to bank A and receive the interbank market return of $\frac{D_2}{D_1}$, it is not able to pay its depositors fully at $t = 2$ and defaults. Since banks B and C lend to bank A, the central bank does not lend to bank A in equilibrium. Its guarantee simply coordinates the lending by banks B and C.

Proposition 7. *If the central bank guarantees to be lender of last resort at a return of $\frac{D_2}{D_1}$, there is a unique first best equilibrium in which banks B and C lend fully to bank A and the central bank does not lend in equilibrium.*

Proof. See appendix. ∎

The role of the central bank as lender of last resort is particular. General liquidity injected by the central bank into the interbank market is not sufficient because this money would not necessary be lent by banks B and C to bank A. Rather, the central bank must commit to lend to bank A directly. Finally, the central bank must offer to lend to bank A at the market interest rate of $\frac{D_2}{D_1}$. Lending at above-market rates does not necessarily resolve the banking crisis.[23] Bank A may not be able to repay the loan at higher rates without defaulting.

7.3. Interbank Market Crisis after September 11, 2001

The banking crisis model may correspond to evidence of the banking crisis and breakdown in interbank lending in the U.S. after September 11, 2001. During the first few days, some banks in Lower Manhattan were not able to make normal daily interbank payments to other banks but were still receiving payments to their Federal Reserve accounts, which created a large displacement among bank balances held at the Federal Reserve. McAndrews and Potter (2002) examine empirical evidence on interbank payments for the two weeks following September 11 to show that even after bank payment systems and interbank payment volumes had recovered by September 14, the timing of bank payments was still delayed within the day due to a breakdown in the coordination of interbank payments. McAndrews and Potter (2002) also suggest that the coordination of interbank lending may have been hampered. There is evidence that lending by at least some large banks appeared to be lower than usual for days while many banks were in need of borrowing funds.[24] McAndrews and Potter (2002) predict that based on empirical evidence of payments timing, banks who could lend would still be reluctant to do so when there is lack of coordination in the interbank payment system. However, the lack of lending for these banks was not necessarily due to physical infrastructure issues. Physical infrastructure damage was the worst on September 11. By September 14, interbank payment volumes had recovered to their pre-September 11 levels, and the bank whose systems were most damaged (Bank of New York) declared "virtually all of its systems are up and running."[25] Moreover, banks can lend excess balances by issuing telephone instructions to the Federal Reserve without the requirement of computer systems.

The evidence raises questions of why, beyond physical infrastructure problems, the level of lending in the interbank market may be hindered in this type of situation. The banking crisis model shows how an initial displacement of bank balances can lead to breakdowns in interbank lending. While the model is based on an initial shock in the fraction of early consumers, any shock that leads to uneven balances among banks is equivalent and gives the same results. The model shows that interbank lending requires coordination. An exogenous shock may have

[23] A growing literature focuses specifically on the issue of whether a central bank should act as lender of last resort and, if so, whether the central bank should lend at, above, or below market rates, dating back to Bagehot (1873). See for instance Diamond and Rajan (2003a), Freixas et al. (2003) and Rochet and Vives (2003). Since the mid-1960s, the U.S. Fed lent funds to banks in need at the discount rate typically set at a discount to the market (federal funds) target rate. The Fed scrutinized the banks that borrowed at the discount rate, which discouraged their use. Starting January 2003, the Fed changed policy to lend funds at above market rates and without scrutiny.

[24] Based on information from conversations with James McAndrews.

[25] McAndrews and Potter (2002)

broken the coordinated lending patterns. For example, additional evidence suggests that some large banks that were typically net lenders before September 11 became net borrowers for several days.[26] The model demonstrates that when lending and payment coordination breaks down, banks that would typically be lending actually lose the excess reserves to lend. Banks can only lend and make payments out of the payments and borrowing received from other banks. The size of payment and lending flows is endogenous and when coordination stops, banks lose the excess funds to lend.

The banking crisis model shows that a central bank lender of last resort is necessary to restore the market and that general liquidity is not necessarily sufficient. Consistent with the model, evidence suggests that the Fed needed to act as lender of last resort. The Fed lent in the initial days $45 billion through the discount window before later injecting even more in general liquidity through open market operations. McAndrews and Potter (2002) argue based on empirical evidence that Federal Reserve open market operations increasing general market liquidity without targeting individual illiquid banks directly through the discount window would not have been sufficient to restore interbank lending. In addition, banks had $4 billion in overnight overdrafts with the Fed in the initial days, which is implicit lender of last resort lending by the Fed.

While the Fed lent through the discount window and injected liquidity through open market operations in unprecedented sums, the coordination of interbank payments and lending was apparently not fully resolved until the end of the second week. The model may also explain this because evidence suggests that the ability of the Fed to guarantee to act as lender of last resort may have been constrained due to initial uncertainty and reluctance by banks to borrow from the discount window. There may have been initial uncertainty regarding the extent to which the Fed would act as a guaranteed lender of last resort. Historically, the discount window lent at below market rates and used scrutiny to discourage over-usage by banks. Borrowing from the discount window was considered a privilege for banks who were approved by the Fed, rather than a standing facility for unlimited borrowing by banks. Indeed, there are moral hazard and judgement difficulties involved with any central bank openly committing to act as a lender of last resort since liquidity versus solvency problems are difficult to discern. The Fed did assure all banks on September 11 and after that the discount window was open for lending, and the Fed took extraordinary efforts to make discount window lending available to banks. However, these assurances were not in place prior. Furthermore, Fed Chairman Alan Greenspan was in Switzerland on September 11 and was not able to return to the U.S. and provide personal assurances immediately.[27] The lack of a previously guaranteed lender of last

[26] Based on information from conversations with James McAndrews.
[27] See Financial Times (2001).

resort could explain why interbank lending would break down. According to the model, the interbank market needs to know in advance that the central bank guarantees to act as lender of last resort in order for interbank lending to be coordinated.

Secondly, there is evidence consistent with the possibility that banks were reluctant to borrow from the Fed through the discount window, hampering the Fed's ability to guarantee to be lender of last resort using the discount window facility in order to restore interbank lending. The $4 billion in overnight overdrafts with the Fed in the initial days suggests that some banks may have preferred to be overdrawn with the Fed overnight rather than to borrow through the discount window. Furthermore, several banks did not repay federal funds loans to other banks immediately after September 11. Again, the ability to borrow from the Fed through the discount window to cover overnight overdrafts and to repay interbank loans would be possible despite infrastructure damage since banks could perform these actions with telephone instructions.

There has traditionally been a reluctance by banks to borrow through the discount window. Peristiani (1998) shows that large banks became very reluctant to borrow from the discount window in the 1980s due to market stigma. The Fed informed banks that it was lending freely after September 11. However, even if banks were willing to borrow to meet their own overdrafts, this is not necessarily sufficient amounts of borrowing to sustain the interbank lending market. If banks were not prepared to borrow from the discount window in order to lend to others, as banks typically borrow in order to lend throughout the day in the interbank market, interbank lending could dry up. In addition, there is evidence of the difficulty in reversing the reluctance of banks to borrow from the Fed. In 2003, the Fed implemented a new policy of lending freely through the discount window at above market rates but with no scrutiny to all healthy banks. This was in part an attempt to remove the stigma of discount window borrowing. However, Furfine (2003) shows evidence of continued reluctance to borrow through the discount window in 2003. This may suggest that banks could have been reluctant to borrow after September 11. As the model demonstrates, banks' reluctance to borrow from the Fed could explain the breakdown of interbank lending. Interbank lending was restored over a two-week period. This can be explained by the model as the market being assured over time that either no bank would default or that the Fed would indeed fully lend to any bank in need.

8. Conclusion

A major theme in the modern banking literature is investigating causes of bank fragility. This paper has studied the reasons for banking crises in a model of modern banking systems with the realistic features of money, nominal contracts and interbank lending. The focus is on withdrawals made with electronic payments since contemporary bank runs typically occur when large depositors transfer large sums by wire. Under the benchmark model with electronic payments and efficient lending among banks, bank runs due to depositor withdrawals do not occur. A banking crisis can only happen if the lending among banks is interrupted. Simply allowing for leakage through currency withdrawals also does not produce runs. Hence, it is important to focus on the interbank market to examine causes of banking crises. If lending in the interbank market is not efficient, banking crises may occur. An exogenous shock to the banking system may lead to a coordination failure among banks, in which those with excess balances do not lend to a bank in need. This paper also shows what factors are not sufficient for bank runs. The potential for all depositors to simultaneously withdraw currency from the bank does not necessarily imply these type of runs will occur. All banks must be illiquid and suspected of being run for Diamond-Dybvig runs that require leakage through currency hoarding to occur. Additionally, aggregate shocks as in Allen and Gale (1998) do not cause bank runs. The price mechanism in the goods market works to allocate goods efficiently among periods between early and late consumers.

Understanding the precise causes of banking crises is important to implementing governmental policy designed to prevent them. The analysis combined with contemporary empirical evidence suggests that in a modern world, concern over bank runs should be more directed toward interbank market crises than depositor runs. Deposit insurance has been a large policy focus to protect banks from crises. However, I have argued that deposit insurance does not protect banks from all crises and likely does not protect banks from the most important type of crises.

By focusing on modern bank payment systems, I have highlighted the interbank market as a major risk of modern banking fragility. The important policy focus is the role of the central bank as lender of last resort. Goodfriend and King (1988) claim there is no need for a lender of last resort. They argue that the central bank should limit its stabilizing role in the financial system to providing general liquidity. However, I provide theoretical justification to support the empirical evidence that a lender of last resort was necessary to resolve the banking crisis of September 11, 2001. Direct lending by the Federal Reserve was needed not just because of physical infrastructure damage that limited the interbank payments system, but also in order to recoordinate lending among banks. The extent of the central bank's role as lender of last resort is an important question for current Fed discount window policy. The recent policy by the Fed, as of 2003, to lend through the discount window at above market rates without scrutiny is an important step to try to remove market stigma and bank reluctance to borrow from the Fed for liquidity needs. However, initial evidence questions the ability of the new discount window policy to allow the Fed to act as a perfect lender of last resort. Furfine (2003) shows evidence that there is a continued reluctance by banks to borrow as only an estimated

2% of the amount of discount window borrowing that would be expected under the new facility has occurred. I show the importance for banks to be willing to freely borrow from the Fed in order for the Fed to be a credible lender of last resort during a liquidity breakdown in the interbank market.

This paper has made several implicit assumptions for simplicity. Relaxing these various assumptions points toward future research. Potential government solutions may then be more thoroughly evaluated. For instance, Skeie (2003a) examines the impact of gross versus net clearinghouse payment systems and reserve requirements. Other assumptions to study include binding limits or charges on intraday overdraft balances for banks; the extent to which inflation and central bank monetary and banking policy effect efficient interbank market interest rates, lending, and bank stability; the effect of bank competition and market power on bank stability; and the extent to which "sticky prices" disrupt market clearing. These paths may lead to more subtle conditions for understanding bank fragility in the context of the methodology introduced in this paper.

Appendix

Proof of Lemma 1. The market clearing price of invested goods in terms of stored goods at $t = 1$ is $P_1 = \frac{(1-\lambda)(1-\alpha^*)}{\lambda\alpha^*}$. Consumption is given by

$$\begin{aligned} C_1 &= 1-\alpha^* + \frac{(1-\lambda)(1-\alpha^*)}{\lambda} = \frac{1-\alpha^*}{\lambda} = C_1^* \\ C_2 &= \alpha^* R + \frac{\lambda\alpha^* R}{1-\lambda} = \frac{\alpha^* R}{1-\lambda} = C_2^*. \end{aligned}$$

The trade is always incentive compatible for late consumers (for any CRRA) since the value of invested goods received is greater than the value of stored goods paid:

$$\frac{(1-\alpha^*)R}{P_1} = \lambda C_2^* > \lambda C_1^* = (1-\alpha^*).$$

The trade is incentive compatible for early consumers if

$$P_1\alpha^* = \frac{C_1^*\alpha^* R}{C_2^*} \geq r\alpha^*,$$

for which CRRA greater than one is sufficient. ■

Proof of Lemma 2.

Lemma 2.1. *$D_{1,2}^{ff} \in \left[1, \frac{D_2}{D_1}\right]$ and $D_{1,2} \in \left[1, \frac{D_2}{D_1}\right]$.*

Proof of Lemma 2.1. Let $L_1^B \widehat{D}_{1,2}^{ff}$ be the maximum feasible amount the bank can repay without defaulting. The maximum feasible interest the bank can pay, $L_1^B \widehat{D}_{1,2}^{ff} - L_1^B$, is given by the total amount of repayment the bank receives from the entrepreneurs minus its total payments to consumers over both periods, which is

$$\min\{K_1^S, Q_1^S P_1\} + \min\{K_2^L, Q_1^L P_1 D_{1,2} + Q_2^L P_2\} - \lambda^w D_1 - (1-\lambda^w)D_2 \leq (\lambda^w - \lambda)(D_2 - D_1).$$

Thus, $(\lambda^w - \lambda)(D_2 - D_1)$ is an upper limit on interest.

If $K_1^S > Q_1^S P_1$, the short term entrepreneur defaults, so $Q_1^S = \beta_0^S = \lambda D_1$ and the loan required is

$$L_1^B = \lambda^w D_1 - Q_1^S P_1 = (\lambda^w - \lambda P_1)D_1.$$

The maximum feasible return on the loan $\widehat{D}_{1,2}^{ff}$ is capped by one plus the interest cap:

$$\widehat{D}_{1,2}^{ff} \leq 1 + \frac{(\lambda^w - \lambda)(D_2 - D_1)}{(\lambda^w - \lambda P_1)D_1}.$$

$K_1^S > Q_1^S P_1$ is equivalent to $\lambda D_1 > \lambda D_1 P_1$ and implies $P_1 < 1$. This implies the maximum feasible return on the loan is less than $\frac{D_2}{D_1}$:

$$\widehat{D}^{ff}_{1,2} < \frac{D_2}{D_1}.$$

If $K^S_1 \leq Q^S_1 P_1$, the loan required is

$$\lambda^w D_1 - K^S_1 = (\lambda^w - \lambda) D_1.$$

$\lambda^w D_1 - K^S_1 = (\lambda^w - \lambda) D_1$.With the interest cap, the maximum feasible return on the loan is less than or equal to $\frac{D_2}{D_1}$:

$$\widehat{D}^{ff}_{1,2} \leq \frac{(\lambda^w - \lambda) D_1 + (\lambda^w - \lambda)(D_2 - D_1)}{(\lambda^w - \lambda) D_1} = \frac{D_2}{D_1}.$$

Thus, $D^{ff}_{1,2} \in \left[1, \widehat{D}^{ff}_{1,2}\right]$, so $D^{ff}_{1,2} \in \left[1, \frac{D_2}{D_1}\right]$. If the loan is made, the second bank pays the full return to its depositors since it is competitive, so $D_{1,2} = D^{ff}_{1,2} \in \left[1, \frac{D_2}{D_1}\right]$.

Lemma 2.2. *If the supply and demand for money are not both zero at* $t = 2$*, then* $P_2 = \overline{P}_2 \equiv \frac{S_2}{\overline{Q}^L_2}$ *and* P_2 *is finite. If* $S_2 = M^L_2 = 0$*,* P_2 *is undetermined and* $Q^L_2 = 0$*. Separately, the quantity of money demanded is greater than or equal to the quantity of money supplied for goods at* $t = 2$*,* $M^L_2 \geq S_2$*, so* $K^L_2 \geq Q^L_1 P_1 D_{1,2} + Q^L_2 P_2$ *and the long term entrepreneur's debt repayment constraint (4.10b) is never slack.*

Proof of Lemma 2.2. Consider the long term entrepreneur's maximization at $t = 1$, (4.10). θ^L_1, θ^L_2, θ^L_3 and θ^L_4 are the Lagrange multipliers associated with the constraints (4.10b) through (4.10e). The necessary Kuhn-Tucker conditions are:

$$-1 + \theta^L_1 - \theta^L_2 \leq 0 \qquad (= 0 \text{ if } Q^L_2 > 0) \quad (8.1a)$$

$$1 - \theta^L_1 \frac{P_1}{P_2} D_{1,2} I_{[M^L_2 \leq P_2 \overline{Q}^L_2]} - \theta^L_1 I_{[M^L_2 > P_2 \overline{Q}^L_2]} + \theta^L_2 - \theta^L_4 \leq 0 \qquad (= 0 \text{ if } \beta^L_1 > 0) \quad (8.1b)$$

$$\begin{aligned} -R + \theta^L_1 r \frac{P_1}{P_2} D_{1,2} I_{[M^L_2 \leq P_2 \overline{Q}^L_2]} + \theta^L_1 I_{[M^L_2 < P_2 \overline{Q}^L_2]} & \qquad\qquad (8.1c) \\ -\theta^L_2 R - \theta^L_3 + \theta^L_4 r \leq 0 & \qquad (= 0 \text{ if } \gamma^L_1 > 0) \end{aligned}$$

where $I_{[\cdot]}$ is the indicator function. The derivative of the constraint (4.10b) is not defined where it binds. For purposes of the Kuhn-Tucker conditions, if the derivative at this point is defined as either the right-hand derivative or the left-hand derivative, the correct solution holds. Thus, for simplicity, rather than transform the problem such that the function is fully differentiable but loses the economic interpretation, I define the derivative at $M^L_2 = P_2 \overline{Q}^L_2$ as equal to the derivative at $M_2^L < P_2\overline{Q}_2^L$. The equivalent holds for functions in which the derivative is not defined everywhere in proofs below in which I use the indicator function.

Consider the short term entrepreneur's maximization at $t = 1$, (4.11). θ_1^S and θ_2^S are the Lagrange multipliers associated with the constraints (4.11b) through (4.11c). The necessary Kuhn-Tucker conditions are:

$$1 - \frac{P_1}{P_2} D_{1,2} I_{[K_1^S \leq Q_1^S P_1]} - \theta_1^S - \theta_2^S \leq 0 \qquad (= \ 0 \text{ if } \beta_1^S > 0) \tag{8.2a}$$

$$\frac{P_1}{P_2} D_{1,2} I_{[K_1^S \leq Q_1^S P_1]} + \theta_1^S - \theta_1^S I_{[K_1^S > \beta_0^S P_1]} + \theta_2^S \leq 0 \qquad (= \ 0 \text{ if } \beta_0^S > 0), \tag{8.2b}$$

I will first show how the relationship between S_2 and M_2^L implies the value of P_2.

If $S_2 > M_2^L$, a marginal dollar is worthless to either the early or late entrepreneur or the consumer. If the price of a dollar in terms of goods is zero, the price of goods in terms of money is infinite, so $Q_2^L = 0$. This is confirmed by the long term entrepreneur's Kuhn-Tucker conditions. Suppose $Q_2^L > 0$. This implies by (8.1a) that $\theta_1^L = 1 + \theta_2^L$. But $M_2^L < S_2 = Q_2^L P_2$ implies by complementary slackness that $\theta_1^L = 0$, which is a contradiction, so $Q_2^L = 0$. If $S_2 > 0$, $P_2 = \frac{S_2}{0} = \infty$. If $S_2 = 0$, P_2 is undefined.

If $S_2 < M_2^L$, the long term entrepreneur defaults, which implies that the long term entrepreneur must sell all her goods at $t = 2$ so $Q_2^L = \overline{Q}_2^L$, and $P_2 = \overline{P}_2$.

If $S_2 = M_2^L = 0$, $Q_2^L = 0$. Suppose not. $Q_2^L > 0$ implies $P_2 = 0$ and by (8.1a) that $\theta_1^L = 1 + \theta_2^L$. Since θ_3^L is the shadow price of $(\beta_0^S + \gamma_1^L - \alpha)$, implying $\theta_3^L < \infty$, (8.1c) implies $P_1 = 0$. But $P_1 = \frac{D_1}{Q_1^S + Q_1^L}$, implying $D_1 = 0$, a contradiction. Thus $Q_2^L = 0$ and P_2 is undefined.

If $S_2 = M_2^L > 0$, $Q_2^L = \overline{Q}_2^L$ as well. Suppose not, $Q_2^L < \overline{Q}_2^L$. If $P_2 \leq \overline{P}_2 \equiv \frac{S_2}{\overline{Q}_2^L}$, the dollars received by the long term entrepreneur are less than the dollars she owes on her loan, $Q_2^L P_2 \leq S_2 \frac{Q_2^L}{\overline{Q}_2^L} < S_2 = M_2^L$, which means the long term entrepreneur defaults, implying $Q_2^L = \overline{Q}_2^L$, a contradiction.

Suppose next that $P_2 > \overline{P}_2$. Consider the aggregate demand schedule $Q_2^D(P_2)$ submitted at $t = 2$ by late buyers, the late consumers and perhaps the short term entrepreneur who purchase goods at $t = 2$. If any individual late buyer demands an amount of goods that is less than he can afford (less by any $\epsilon > 0$ amount) at the price $P_2 > \overline{P}_2$, in effect supplying less money for goods than he has available, the total dollars received by the long term entrepreneur is less than M_2^L for any supply schedule submitted by the long term entrepreneur. Mathematically, $Q_2^D(P_2) < \frac{S_2}{P_2}$ implies $Q_2^L P_2 < S_2 = M_2^L$. This is again a default by the long term entrepreneur, implying $Q_2^L = \overline{Q}_2^L$. But this implies $P_2 = \frac{S_2}{\overline{Q}_2^L} = \overline{P}_2$, a contradiction to $P_2 > \overline{P}_2$. For any supply schedule submitted by the long term entrepreneur consistent with the requirement that she supplies $Q_2^L = \overline{Q}_2^L$ when she defaults on her loan, P_2 greater than $\overline{P}_2$ cannot be a market clearing price. Since every individual late buyer strictly favors submitting the above demand schedule for $P_2 > \overline{P}_2$, this is the demand schedule that is submitted. Thus, $Q_2^L = \overline{Q}_2^L$, and $P_2 = \overline{P}_2$.[28]

Thus, I have shown that $S_2 = M_2^L = 0$ implies P_2 is undetermined. I have also shown that $M_2^L = S_2 > 0$ and $M_2^L > S_2$ implies $P_2 = \overline{P}_2$.

Second, I will show that $M_2^L \geq S_2$ and $K_2^L \geq Q_1^L P_1 D_{1,2} + Q_2^L P_2$. Suppose not, $M_2^L < S_2$. This implies $Q_2^L = 0$.

I will show that $M_2^L < S_2$ implies $\lambda^p = 1$. Suppose not, $\lambda^p < 1$. If $S_2 = 0$, $\lambda^p = 1$, which is a contradiction.

If $S_2 > 0$, $P_2 = \infty$. Suppose $P_1 = \infty$ as well. This implies that marginal dollars are worthless at $t = 1$ and $t = 2$. This implies the dollars demanded by entrepreneurs at each period are less than the dollars supplied by consumers at each period, so the total dollars demanded in both periods are less than the total dollars supplied in both periods:

$$
\begin{aligned}
K_1^S + K_2^L &< \lambda^p D_1 + (1-\lambda^w) D_2 + (\lambda^w - \lambda^p) D_1 D_{1,2} \\
\lambda D_1 + (1-\lambda) D_2 &< \lambda^p D_1 + (1-\lambda^w) D_2 + \lambda^w D_2 - \lambda^p D_2 \\
&< \lambda^p D_1 + (1-\lambda^p) D_2 \\
&\leq \lambda D_1 + (1-\lambda) D_2,
\end{aligned}
$$

which is a contradiction.

So suppose instead P_1 is finite. The late consumer's optimization problem shows that $\frac{D_1}{P_1} > \frac{D_2}{P_2} = 0$ implies $\lambda^p = 1$, which is a contradiction. Thus, $\lambda^p = 1$.

Now I will show that $M_2^L < S_2$ implies $S_2^S = 0$. Suppose not, $S_2^S > 0$, so $Q_1^S P_1 > K_1^S$. From the short term entrepreneur's optimization, complementary slackness of θ_1^S implies $\theta_1^S = 0$. Suppose $\beta_1^S = \beta_0^S$. This implies $Q_1^S = 0$ and $K_1^S > Q_1^S P_1 = 0$, which is a contradiction. Thus $\beta_1^S < \beta_0^S$ and $\theta_2^S = 0$ by complementary slackness. Substituting into (8.2a) implies $1 \leq 0$, which is a contradiction, so $S_2^S \not> 0$. Since $S_2^S \geq 0$ by definition, $S_2^S = 0$, and $K_1^S \geq Q_1^S P_1$.

Now I will show $S_2^S = 0$ and $\lambda^p = 1$ imply a contradiction. $S_2^S = 0$ and $\lambda^p = 1$ imply $S_2 = 0$, so $M_2^L < 0$, and $K_2^L < Q_1^L P_1 D_{1,2}$. $S_2^S = 0$ implies $Q_1^S P_1 \leq K_1^S$, so the bank needs a

[28] This determination of $P_2 = \overline{P}_2$ relies on the assumption of a finite number of consumers who are not price takers. However, the problem of price determination is a common problem in general equilibrium, and this is just one of several possible techniques in this model to resolve the price determination problem such that the long term entrepreneur sells $\overline{Q}_2^L$ at $t = 2$ so that $P_2 = \overline{P}_2$. It is economically intuitive that purely competitive entreprenuers sell all of their goods and break even rather than keep a monopolist-type profit. Another technique to arrive at this result is if the bank held and invested $\epsilon > 0$ goods at $t = 0$ and sold all goods at $t = 2$. The long term entrepreneur would have to sell $Q_2^L = \overline{Q}_2^L$ to repay K_2^L.

loan of

$$L_1^B = D_1 - Q_1^S P_1 = D_1 - Q_1^S \frac{D_1}{Q_1^S + Q_1^L} = Q_1^L P_1.$$

The maximum feasible amount the bank can repay is $L_1^B \widehat{D}_{1,2}^{ff} = K_2^L$, which it can always repay since the long term entrepreneur does not default. The loan is always granted if $L_1^B \widehat{D}_{1,2}^{ff} \geq L_1^B$. If $P_1 \geq 1$, the short term entrepreneur does not default, so by $S_2^S = 0$, $Q_1^S P_1 = K_1^S$.

$$L_1^B = D_1 - Q_1^S P_1 = D_1(1 - \lambda P_1) \leq D_1(1 - \lambda) < D_2(1 - \lambda) = K_2^L,$$

so the loan condition holds. If $P_1 < 1$,

$$Q_1^L P_1 < Q_1^L \leq \alpha r < \alpha R = K_2^L,$$

so the loan condition holds. Since $D_{1,2}^{ff} \leq \widehat{D}_{1,2}^{ff}$, $D_{1,2} = D_{1,2}^{ff}$ and $\widehat{D}_{1,2}^{ff} = D_{1,2}^{ff}$ imply $D_{1,2} \leq D_{1,2}^{ff}$, or $Q_1^L P_1 D_{1,2} \leq K_2^L$, which is a contradiction to $K_2^L < Q_1^L P_1 D_{1,2}$.

Hence, I have shown $M_2^L \geq S_2$, which implies that (4.10b) is not slack since $Q_2^L \leq \overline{Q}_2^L$. Since

$$M_2^L = K_2^L - Q_1^L P_1 D_{1,2} \geq S_2 = Q_2^L P_2,$$

I have also shown $K_2^L \geq Q_1^L P_1 D_{1,2} + Q_2^L P_2$.

Third, I will show that $P_2 \neq \infty$. Suppose not, $P_2 = \infty$.

Suppose also $Q_2^L > 0$. $M_2^L < Q_2^L P_2 = \infty$ implies $\theta_1^L = 0$ by complementary slackness. By (8.1a), $Q_2^L > 0$ implies $1 + \theta_2^L = \theta_1^L = 0$, which is a contradiction. Thus, $Q_2^L = 0$.

Suppose $M_2^L = S_2 = 0$. This implies P_2 is undefined, which is a contradiction.

Suppose instead $S_2 \leq M_2^L$ and $M_2^L > 0$. This implies $Q_2^L = \overline{Q}_2^L$, which implies $\overline{Q}_2^L = (\alpha - \gamma_1^L)R + \beta_1^L = 0$, or $\beta_1^L = 0$ and $\gamma_1^L = \alpha$. Consider the long term entrepreneur's Kuhn-Tucker conditions. $\gamma_1^L r = \alpha r > \beta_1^L = 0$, which implies by complementary slackness that $\theta_4^L = 0$. (8.1a) implies $\theta_1^L \leq 1 + \theta_2^L$. This implies by (8.1b) that $1 \leq 0$, which is a contradiction. Thus, I have shown $P_2 \neq \infty$.

Proof of Lemma 2 (continued). From Lemma 2.2,

$$S_2 \leq M_2^L = K_2^L - Q_1^L P_1 D_{1,2},$$

which can be rewritten as $Q_2^L P_2 + Q_1^L P_1 D_{1,2} \leq K_2^L$. So the condition for the bank to not default (4.12) is

$$Q_2^L P_2 + Q_1^L P_1 D_{1,2} \geq L_1^B D_{1,2} + (1 - \lambda^w) D_2.$$

If $K_1^S \leq Q_1^S P$, the short term entrepreneur does not default and the loan to the bank is

$L_1^B = (\lambda^w - \lambda)D_1 D_{1,2}$. Since $Q_2^L P_2 = S_2$, (4.12) is equivalent to

$$\begin{aligned}
Q_2^L P_2 + Q_2^L P_1 D_{1,2} &\geq L_1^B D_{1,2} + (1-\lambda^w)D_2 \\
(Q_1^S P_1 - K_1^S)D_{1,2} + Q_1^L P_1 D_{1,2} &\geq (\lambda^p - \lambda)D_1 D_{1,2} \\
[(\lambda^p - \lambda)D_1 - (Q_1^S + Q_1^L)\frac{\lambda^p D_1}{Q_1^S + Q_1^L} + \lambda D_1]D_{1,2} &\leq 0 \\
0 &\leq 0.
\end{aligned}$$

Thus the bank does not default for all λ^w and λ^p.

If $K_1^S > Q_1^S P_1$, the short term entrepreneur defaults and must sell all goods so $Q_1^S = \beta_0^S = \lambda D_1$. This implies the loan to the bank is $L_1^B = (\lambda^w D_1 - Q_1^S P_1)$, so (4.12) is equivalent to

$$\begin{aligned}
Q_2^L P_2 + Q_1^L P_1 D_{1,2} &\geq L_1^B D_{1,2} + (1-\lambda^w)D_2 \\
(\lambda^w - \lambda^p)D_1 D_{1,2} + Q_1^L P_1 D_{1,2} &\geq (\lambda^w D_1 - Q_1^S P_1)D_{1,2} \\
[\lambda^w D_1 - (Q_1^S + Q_1^L)\frac{\lambda^p D_1}{Q_1^S + Q_1^L} - (\lambda^w - \lambda^p)D_1]D_{1,2} &\leq 0 \\
0 &\leq 0.
\end{aligned}$$

Thus the bank does not default for all λ^w and λ^p. Since the bank never defaults and $D_{1,2} \geq 1$, the second bank always grants the loan L_1^B. ■

Proof of Proposition 1. I show that at $t = 1$, the short term entrepreneur sells all of his goods and the long term entrepreneur does not liquidate any goods: $\beta_1^S = \beta_1^L = \gamma_1^L = 0$.

Suppose $\beta_1^L > 0$. (4.10e) implies $\gamma_1^L > 0$. (8.1c) and (8.1b) imply $r \geq R$, which is a contradiction to the assumption that $r < 1 < R$. Thus $\beta_1^L = 0$, and $\gamma_1^L > 0$ implies $\theta_4^L = 0$ by complementary slackness.

$S_2 < M_2^L$ implies $Q_2^L P_2 = S_2 < M_2^L$, so $\theta_1^L = 0$ and the long term entrepreneur defaults, so $Q_2^L = \overline{Q}_2^L$. $S_2 = M_2^L$ implies either $Q_2^L = \overline{Q}_2^L$ or $Q_2^L = 0$, as shown above.

Suppose $\gamma_1^L = \alpha$. This implies $Q_2^L = 0$, which implies either $P_2 = \infty$, a contradiction to Lemma 2.2, or $S_2 = 0$. Suppose $S_2 = 0$. This implies $\lambda^p = 1$. The late consumer's optimization implies $P_1 \leq \frac{D_1}{D_2} P_2 < P_2$, which by (8.1c) implies

$$(1+\theta_2^L)R + \theta_3^L \leq (1+\theta_2^L)\frac{D_1}{D_2} D_{1,2} r \leq (1+\theta_2^L)r.$$

This implies $R \leq r$, which is a contradiction to the assumption that $r < 1 < R$. Thus $\gamma_1^L < \alpha$, and $\theta_3^L = 0$ by complementary slackness.

Suppose $\beta_1^S = \beta_0^S$. This implies $Q_1^S = 0$, so $K_1^S = \lambda D_1 > Q_1^S P_1$, which means the short term entrepreneur defaults and must sell all goods at $t = 1$, so $Q_1^S = \beta_0^S > 0$, which is a contradiction. Thus $\beta_1^S < \beta_0^S$, and $\theta_2^S = 0$ by complementary slackness.

Suppose $\beta_1^S > 0$ and $\gamma_1^L > 0$. Complementary slackness implies $\theta_4^L = 0$. Suppose $M_2^L > \overline{Q}_2^L P_2$. Then (8.1b) implies $1 \leq 0$, a contradiction. Suppose instead $M_2^L \leq \overline{Q}_2^L P_2$. $\beta_0^S > 0$ implies $Q_1^S P_1 \geq K_1^S$. (8.2a) and (8.1c) imply

$$P_2 = P_1 D_{1,2} + \theta_1^S P_2 = P_1 D_{1,2} \frac{r}{R},$$

or $\theta_1^S P_2 = P_1 D_{1,2} \left(\frac{r}{R} - 1\right) < 0$, which is a contradiction since P_2 and θ_1^S cannot be negative. Thus $\beta_1^S > 0$ implies $\gamma_1^L = 0$, and $\gamma_1^L > 0$ implies $\beta_1^S = 0$ and $\theta_4^L = 0$.

Suppose $\beta_1^S > 0$. Consider the short term entrepreneur's debt constraint given by (4.11b). Suppose $K_1^S > Q_1^S P_1$ and the short term entrepreneur defaults. This implies $Q_1^S = \beta_0^S$, so $\beta_1^S = 0$, which is a contradiction. Suppose next that $K_1^S < Q_1^S P_1$, which implies $\theta_1^S = 0$. (8.2a) implies $P_2 = P_1 D_{1,2} \leq P_1 \frac{D_2}{D_1}$, or $\frac{D_2}{P_2} \geq \frac{D_1}{P_1}$. The late consumer's optimization implies $\lambda^p = \lambda$, so

$$Q_1^S P_1 = (\lambda D_1 - \beta_1^S) \frac{\lambda D_1}{\lambda D_1 - \beta_1^S} = K_1^S,$$

which is a contradiction. Suppose finally that $K_1^S = Q_1^S P_1$. Writing this as

$$\lambda D_1 = (\lambda D_1 - \beta_1^S) \frac{\lambda^p D_1}{(\lambda D_1 - \beta_1^S)} = \lambda^p D_1,$$

this implies $\lambda^p = \lambda$. $\gamma_1^L = 0$ implies $Q_1^L = 0$ and $M_2^L = K_2^L = (1-\lambda) D_2 = S_2 > 0$, thus $Q_2^L = \overline{Q}_2^L = (1-\lambda) D_2$ from Lemma 2.2, so $P_2 = 1$. (8.2a) implies $\frac{1-\theta_1^S}{P_1} = \frac{1}{P_2} D_{1,2}$, or

$$\frac{(\lambda D_1 - \beta_1^S)(1 - \theta_1^S)}{\lambda D_1} = D_{1,2} \geq 1,$$

which implies $\beta_1^S \leq 0$, a contradiction. Thus $\beta_1^S = 0$.

Suppose $\gamma_1^L > 0$. $\theta_4^L = 0$, so (8.1b) implies $M_2^L \leq \overline{Q}_2^L P_2$. (8.1c) implies $P_1 \theta_1^L D_{1,2} r = P_2 (1 + \theta_2^L) R$, or since $\theta_1^L = 1 + \theta_2^L$ by complementary slackness of (8.1a),

$$P_2 R = P_1 D_{1,2} r \leq \frac{D_2}{D_1} P_1 r.$$

Rewritten, $\frac{D_2}{P_2} r \geq \frac{D_1}{P_1} R$, which implies $\frac{D_2}{P_2} > \frac{D_1}{P_1}$, so by the late consumer's optimization, $\lambda^p = \lambda$. Since $Q_1^L > 0$,

$$M_2^L = (1-\lambda) D_2 - Q_1^L P_1 D_{1,2} < (1-\lambda) D_2.$$

Substituting for Q_1^S and P_1,

$$Q_1^S P_1 = \lambda D_1 \frac{\lambda D_1}{\lambda D_1 + \gamma_1^L r} < \lambda D_1 = K_1^S,$$

so the short term entrepreneur defaults, thus $S_2^S = 0$ and $S_2 = (1-\lambda)D_2$. Hence $S_2 > M_2^L$, which is a contradiction. Thus $\gamma_1^L = 0$.

Finally, $\beta_1^S = \beta_1^L = \gamma_1^L = 0$ is a solution to (8.1), (8.2), and the constraints from (4.10) and (4.11), and gives a maximum for the objective functions in (4.10a) and (4.11a), so it is the unique solution to the short term and long term entrepreneurs' problems (4.10) and (4.11). ■

Proof of Lemma 3.

Lemma 3.1. *If the supply and demand for money are not zero at $t = 2$, then $P_2 = \overline{P}_2 \equiv \frac{S_2}{\overline{Q}_2^L}$. Separately, the demand for money is always equal to the supply of money at $t = 2$, $M_2^L = S_2$, so the entrepreneur's $t = 2$ debt repayment constraint (4.10b) is always binding.*

Proof of Lemma 3.1. Efficiency of interbank lending implies $D_{1,2}^{ff} = 1$. The second bank is competitive so $D_1 = 1$. Since

$$M_2^L = K_2^L + K_1^S - Q_1^L P_1 = D - \lambda^p D = (1-\lambda^p)D$$

and $S_2 = (1-\lambda^p)D,\ S_2 = M_2^L$.

Next, I show that if $S_2 = M_2^L > 0$, $Q_2^L = \overline{Q}_2^L$ and $P_2 = \overline{P}_2 = \frac{S_2}{\overline{Q}_2^L}$. Suppose not, $Q_2^L < \overline{Q}_2^L$. If $P_2 \leq \overline{P}_2 \equiv \frac{S_2}{\overline{Q}_2^L}$, the dollars received by the entrepreneur are less than the dollars she owes on her loan,

$$Q_2^L P_2 \leq S_2 \frac{Q_2^L}{\overline{Q}_2^L} < S_2 = M_2^L,$$

which means the entrepreneur defaults, implying $Q_2^L = \overline{Q}_2^L$, a contradiction. Suppose next that $P_2 > \overline{P}_2$. Consider the aggregate demand schedule $Q_2^D(P_2)$ submitted at $t = 2$ by late buyers, late consumers who buy goods at $t = 2$. If any individual late buyer demands an amount of goods that is less than he can afford (less by any $\epsilon > 0$ amount) at the price $P_2 > \overline{P}_2$, in effect supplying less money for goods than he has available, the total dollars received by the entrepreneur is less than M_2^L for any supply schedule submitted by the entrepreneur. Mathematically, $Q_2^D(P_2 > \overline{P}_2) < \frac{S_2}{P_2}$ implies $Q_2^L(P_2 > \overline{P}_2)P_2 < S_2 = M_2^L$. This is again a default by the entrepreneur at $t = 2$, implying $Q_2^L = \overline{Q}_2^L$. But this implies $P_2 = \frac{S_2}{\overline{Q}_2^L} = \overline{P}_2$, a contradiction to $P_2 > \overline{P}_2$. For any supply schedule submitted by the entrepreneur consistent with the requirement that she supplies $Q_2^L = \overline{Q}_2^L$ when she defaults on her loan, P_2 greater than $\overline{P}_2$ cannot be a market clearing price. With this strategy, $P_2 \leq \overline{P}_2$ and a late buyer receives

$C_2(P_2 \leq \overline{P}_2) = \frac{D}{P_2} \geq \frac{D\overline{Q}_2^L}{S_2}$. Without this strategy, a later buyer receives $C_2(P_2 > \overline{P}_2) = \frac{D}{P_2} < \frac{D\overline{Q}_2^L}{S_2}$. Since every individual late buyer strictly favors submitting an individual demand schedule of this nature for $P_2 > \overline{P}_2$, the above aggregate demand schedule is submitted. Thus, $Q_2^L = \overline{Q}_2^L$, and $P_2 = \overline{P}_2$.

Proof of Lemma 3 (continued). The bank does not default if the repayment it receives from the entrepreneur at $t = 2$ cover the bank's repayment on the loan needed at $t = 1$ plus its payment to late-withdrawing consumers:

$$\min\{K_2^L,\ Q_2^L P_2 + Q_1^S P_1 - K_1^S\} \geq L_1^B D_{1,2}^{ff} + (1 - \lambda^w)D = L_1^B + (1 - \lambda^w)D. \tag{8.3}$$

Since

$$S_2 = M_2^L = K_2^L + K_1^S - Q_1^S P_1$$

and $S_2 = Q_2^L P_2$, it follows that $K_2^L = Q_2^L P_2 + Q_1^S P_1 - K_1^S$, so (8.3) reduces to $(1 - \lambda)\, D \geq L_1^B + (1 - \lambda^w)D$. The loan needed at $t = 1$ is $L_1^B = \lambda^w D - \min\{K_1^S, Q_1^S P_1\}$. Suppose $K_1^S > Q_1^S P_1$. This implies $\lambda D > \lambda^p D$, or $\lambda^p < \lambda$, a contradiction to the assumption that $\lambda^p \geq \lambda$, so $K_1^S \leq Q_1^L P_1$. Thus, $L_1^B = \lambda^w D - \lambda D = (\lambda^w - \lambda)D$, and (8.3) requires

$$(1 - \lambda)\, D \geq (\lambda^w - \lambda)D + (1 - \lambda^w)D = (1 - \lambda)D,$$

which always holds. Thus the bank does not default for all λ^w and λ^p. Since the bank never defaults and $D_{1,2} = 1$, the second bank always grants the loan L_1^B. ■

Proof of Proposition 2. Consider the entrepreneur's maximization at $t = 0$, (5.13). θ_1^S, θ_1^L, θ_2^L and θ_4^L are the Lagrange multipliers associated with the constraints (4.11b), (4.10b), (4.10c) and (5.13b). The necessary Kuhn-Tucker conditions are (8.1a),

$$\rho - \theta_1^S - \theta_1^L \frac{P_1}{P_2} I_{[M_2^L \leq \overline{Q}_2^L P_2]} - \theta_1^L \rho I_{[M_2^L > \overline{Q}_2^L P_2]} + \theta_2^L \rho - \theta_4^L \leq 0 \qquad (= 0 \text{ if } \beta_1^L > 0), \tag{8.4}$$

where $I_{[\cdot]}$ denotes the indicator function.

I show that at $t = 1$, the entrepreneur sells $1 - \alpha - \beta_1^L$, where β_1^L is given by (5.10) and (5.11).

Suppose $\beta_1^L = \beta_0^S$. This implies $Q_1^S = 0$, which implies $P_1 = \infty$. Suppose $P_2 = \infty$ as well. This implies that marginal dollars are worthless at $t = 1$ and $t = 2$. This implies the dollars demanded by entrepreneurs at each period are less than the dollars supplied by consumers at each period, so the total dollars demanded in both periods are less than the total dollars supplied in both periods:

$$K_1^S + K_2^L < \lambda^p D + (1 - \lambda^w)D + (\lambda^w - \lambda^p)D,$$

or $D < D$, which is a contradiction.

So suppose instead P_2 is finite. Since the bank does not default, $P_1 > P_2$ implies $\lambda^p = \lambda$. Consider the aggregate demand schedule $Q_1^D(P_1)$ submitted at $t = 1$ by early buyers, early and late consumers who buy goods at $t = 1$. If any individual early buyer demands an amount of goods that is less than he can afford (less by any $\epsilon > 0$ amount) at the price $P_1 = \infty$, in effect supplying less money for goods than he has available, the total dollars received by the entrepreneur at $t = 1$ is less than K_1^S for any supply schedule submitted by the entrepreneur. Mathematically, $Q_1^D(P_1 = \infty) < \frac{\lambda D}{P_1}$ implies $Q_1^S(P_1 = \infty)P_1 < \lambda D$. This is a default by the entrepreneur at $t = 1$, implying $\beta_1^L = 0$ and $Q_1^S = 1 - \alpha$. But this implies $P_1 = \frac{\lambda D}{\lambda D} = 1$, a contradiction to $P_1 = \infty$. For any supply schedule submitted by the entrepreneur consistent with the requirement that he supplies $Q_1^S = \lambda D$ when he defaults on his loan, P_1 equal to ∞ cannot be a market clearing price. With this strategy, $P_1 < \infty$ and an early buyer receives $C_1(P_1 < \infty) = \frac{D}{P_1} > 0$. Without this strategy, an early buyer receives $C_1(P_1 = \infty) = \frac{D}{P_1} = 0$. Since every individual early buyer strictly favors submitting the above demand schedule for $P_1 = \infty$, this is the demand schedule that is submitted. Thus, $P_1 < \infty$, a contradiction. Hence, $\beta_1^L < \beta_0^S$, and $\theta_4^L = 0$ by complementary slackness.

$\lambda^p < 1$ implies that $S_2 > 0$. From Lemma 2.2, $M_2^L = S_2 > 0$, so $Q_2^L = \overline{Q}_2^L$ and $\theta_1^L = 1 + \theta_2^L$ by complementary slackness.

Suppose $M_2^L > \overline{Q}_2^L P_2$. (4.10b) and (4.10c) imply $Q_2^L = \overline{Q}_2^L$, so $M_2^L > Q_2^L P_2 = S_2$, which is a contradiction to Lemma 3. Thus, $M_2^L \leq \overline{Q}_2^L P_2$.

Suppose $\beta_1^L > 0$. (8.1b) implies $P_1 = P_2(\rho - \frac{\theta_1^S}{1+\theta_2^L})$. Substituting for P_1 and P_2,

$$\frac{\lambda^p D}{1 - \alpha - \beta_1^L} = \frac{(1 - \lambda^p)D}{(\alpha R + \rho\beta_1^L)}(\rho - \frac{\theta_1^S}{1 + \theta_2^L}).$$

Solving for β_1^L,

$$\beta_1^L = \frac{(1 - \lambda^p)\,(1 - \alpha)\,(\rho - \frac{\theta_1^S}{1+\theta_2^L}) - \lambda^p \alpha R}{\rho - (1 - \lambda^p)\left(\frac{\theta_1^S}{1+\theta_2^L}\right)}.$$

Note that $P_1 \geq 0$ implies $\frac{\theta_1^S}{1+\theta_2^L} < \rho$, so the denominator is positive.

Consider $R \geq \overline{R}$. This implies

$$\begin{aligned}
\beta_1^L &\leq \frac{(1-\alpha)\left[\lambda(1-\lambda^p)(\rho - \frac{\theta_1^S}{1+\theta_2^L}) - \lambda^p (1-\lambda)\rho\right]}{\lambda\left[\rho - (1-\lambda^p)\left(\frac{\theta_1^S}{1+\theta_2^L}\right)\right]} \\
&\leq \frac{-(1-\alpha)\lambda(1-\lambda)(\frac{\theta_1^S}{1+\theta_2^L})}{\lambda\left[\rho - (1-\lambda)\left(\frac{\theta_1^S}{1+\theta_2^L}\right)\right]} \\
&\leq 0,
\end{aligned}$$

which is contradiction. Thus, $\beta_1^L = 0$ is a unique solution.

Now consider $R < \overline{R}$, and suppose $\beta_1^L > 0$. The Lagrange multiplier θ_1^S can be written as

$$\theta_1^S = \frac{\partial f^*(\mathbf{c})}{\partial c_1} = \frac{\partial f(\beta_1^{L*}(\mathbf{c}), Q_2^{L*}(\mathbf{c}))}{\partial c_1},$$

where f is the objective function of the entrepreneur's maximization, $f = \overline{Q}_2^L - Q_2^L = \alpha R + \rho\beta_1^L - Q_2^L$, and $\mathbf{c}$ is the vector of constants in the constraints of the entrepreneur's maximization problem, and the superscript asterisk (*) denotes a choice variable is at its optimum as a function of the vector of constants $\mathbf{c}$. Thus $f^*(\mathbf{c}) = f(\beta_1^{L*}(\mathbf{c}), Q_2^{L*}(\mathbf{c}))$ equals the value of the objective function as a function of the choice variables at their optimum value. c_1 is the constant for the constraint given in (4.11b), $c_1 = \beta_0^S - \min\left\{\frac{K_1^S}{P_1}, \beta_0^S\right\}$, since (4.11b) is written formally as $\beta_1^L \leq \beta_0^S - \min\left\{\frac{K_1^S}{P_1}, \beta_0^S\right\}$.

If (4.11b) is not binding, $\theta_1^S = 0$ by complementary slackness. If (4.11b) is binding, $\frac{\partial \beta_1^{L*}(\mathbf{c})}{\partial c_1} = 1$. Since $\lambda^p < 1$, $S_2 > 0$ and $Q_2^L = \overline{Q}_2^L$, so (4.10c) is binding. (4.10c) is written formally as $Q_2^L - \rho\beta_1^L - \alpha R \leq 0$, so

$$\frac{\partial Q_2^{L*}(\mathbf{c})}{\partial c_1} = \frac{\partial Q_2^{L*}(\mathbf{c})}{\partial \beta_1^L} = \rho.$$

Thus,

$$\theta_1^S = \frac{\partial f^*(\mathbf{c})}{\partial c_1} = \rho\frac{\partial \beta_1^{L*}(\mathbf{c})}{\partial c_1} - \frac{\partial Q_2^{L*}(\mathbf{c})}{\partial c_1} = 0.$$

Since $\theta_1^S = 0$, $\beta_1^L = (1-\lambda^p)(1-\alpha) - \frac{1}{\rho}\lambda^p \alpha R > 0$, so $\beta_1^L > 0$ indeed.

By (8.4), $P_1 = \rho P_2$. Thus, $\rho > 1$ implies $P_1 > P_2$, which implies by the late consumers' optimization that $\lambda^p = \lambda$, so $\beta_1^L = (1-\lambda)(1-\alpha) - \frac{1}{\rho}\alpha R$ is the unique solution.

Furthermore, $\rho = 1$ implies $P_1 = P_2$, which implies by the late consumers' optimization that $\lambda^p = \lambda$, so $\beta_1^L = (1-\lambda)(1-\alpha) - \lambda\alpha R$ is the unique solution. Finally, (5.10) and (5.11) is a solution to (8.1a) and (8.4) and the constraints from (5.13), and gives a maximum for the objective function (5.13a), so it is the unique solution to the entrepreneur's problem (5.13). ■

Proof of Proposition 3. For simplicity, I will change the reference of all short term entrepreneur variables are that denoted with the superscript "S" to refer to the central bank for this section. Since the central bank may retire currency at $t = 1$ or $t = 2$, I will refer to its liability as K^S rather than K_1^S. The central bank's demand for money at $t = 1$ is the amount of its currency liabilities, λD_1, plus any money which is spent on goods at $t = 2$. This is the same demand for money as the short term entrepreneur has in the benchmark model.

Consider assumption (A1). Without loss of generality, assume any currency received by the long term entrepreneur, original bank or second bank is first redeemed to the central bank for electronic money or goods at $t = 1$ and $t = 2$ if possible, then alternatively redeemed with the late entrepreneur for electronic money or goods if possible. Let $\lambda^c - \lambda^p$ be the fraction of consumers who are late consumers and withdraw currency at $t = 1$ and hoard it (store the currency outside of a bank) until $t = 2$ when they spend it on goods. Thus $\lambda^w - \lambda^c$ is the fraction of late consumers who withdraw early and redeposit with the second bank, and $\lambda^p \leq \lambda^c \leq \lambda^w$. For a case of currency hoarding with a given λ^c, λ^p and λ^w such that $\lambda^c > \lambda^p$ and $\lambda^c \leq \lambda^w$, let the "equivalent problem" with no currency hoarding refer to a case with the same λ^p and λ^w, but $\lambda^c = \lambda^p$. Thus the amount of hoarding $\lambda^c - \lambda^p$ plus redepositing $\lambda^w - \lambda^c$ in a case with hoarding is equal to the amount of redepositing $\lambda^w - \lambda^p$ in a equivalent case but without hoarding.

Suppose $P_1 \geq 1$ and late consumers do not hoard currency. $P_1 = \frac{\lambda^p D_1}{\lambda D_1 + Q_1^L}$ implies $\lambda^p > \lambda$. At $t = 1$, the central bank receives $Q_1^S P_1 \geq \lambda D_1$, so the central bank receives all outstanding currency and may receive some electronic money, thus the central bank K^S liability is satisfied. The long term entrepreneur receives $Q_1^L P_1 \geq 0$ which is only electronic money. The rest of the model is the same as the benchmark model and the proof is identical.

Suppose $P_1 \geq 1$ and late consumers do hoard currency, and compare the problem to the equivalent problem without hoarding. The central bank now receives at $t = 1$ an additional amount $\lambda^c - \lambda^p$ of electronic money over that of the equivalent non-hoarding case to replace the currency the hoarders hold. This additional amount of electronic money is redeposited by early-withdrawing late consumers in the equivalent problem and is deposited to the second bank by the central bank now. At $t = 2$, hoarders exchange their $\lambda^c - \lambda^p$ in currency for electronic money with the central bank. But the central bank has an additional $(\lambda^c - \lambda^p)(D_{1,2} - 1)$ to spend on goods at $t = 2$ greater than that in the equivalent problem, and hoarders have that much less to spend on goods at $t = 2$. Thus,

$$S_2 = (1 - \lambda^w)D_2 + (\lambda^w - \lambda^c)D_1 D_{1,2} + (\lambda^c - \lambda^p)D_1 + S_2^S D_{1,2},$$

where $S_2^S D_{1,2}$ is given by

$$S_2^S D_{1,2} = [(Q_1^S P_1 - K^S)D_{1,2} + (\lambda^c - \lambda^p)D_1(D_{1,2} - 1)]^+.$$

But the total dollars supplied at $t = 2$,

$$S_2 = (1 - \lambda^w)D_2 + (\lambda^w - \lambda^p)D_1 D_{1,2} + (Q_1^S P_1 - K^S)^+,$$

is unchanged from the equivalent problem. The rest of the proof follows that of the benchmark model with the exception that the late hoarders consume less than by withdrawing at $t = 2$: $\frac{D_1}{P_2} < \frac{D_2}{P_2}$. Since the bank does not default, the hoarders strictly prefer to withdraw at $t = 2$, which implies $\lambda^w = \lambda^p = \lambda$, a contradiction. Thus there is no hoarding and no run.

Suppose $P_1 < 1$ and late consumers do not hoard currency. At $t = 1$, the central bank receives $Q_1^S P_1 < \lambda D_1$, so the central bank receives only currency and the K^S liability is not satisfied. Since the central bank is exhausted of goods and has no electronic money, it does not redeem any additional currency at $t = 2$ and the K^S liability goes unsatisfied. At $t = 1$, the long term entrepreneur receives $Q_1^L P_1 > (\lambda^p - \lambda)D_1$ in currency, and possibly some electronic money. The second bank receives, from the late entrepreneur and any late consumers who withdraw early and redeposit, funds of $(\lambda^w - \lambda P_1)D_1$, which is the same as in the benchmark model. The original bank needs to borrow $(\lambda^w - \lambda)D_1$, which is less than the amount the second bank has to lend. This is the amount the bank borrows for the case of $P_1 \geq 1$. The bank borrows $L_1^B = (\lambda^w - \lambda)D_1$ and repays $L_1^B D_{1,2}^{ff}$, and the second bank pays depositors

$$D_{1,2} = \frac{L_1^B D_{1,2}^{ff}}{(\lambda^w - \lambda P_1)D_1} = \frac{L_1^B D_{1,2}^{ff}}{L_1^B + \lambda(1 - P_1)D_1},$$

which is the same $D_{1,2}$ in the benchmark model for the case of $P_1 < 1$. The simplification condition holds for both amounts:

$$1 \leq D_{1,2} < D_{1,2}^{ff} \leq \frac{D_2}{D_1},$$

as shown by the condition for the bank to not default, (4.12). (4.12) holds since

$$(\lambda^w - \lambda P_1)D_1 > (\lambda^w - \lambda)D_1.$$

The rest of the model is the same as the benchmark model and the proof is identical.

Suppose $P_1 < 1$ and late consumers do hoard currency, and compare this problem to the equivalent problem without hoarding. Any initial amount of currency hoarded is an amount of extra electronic money the long term entrepreneur receives at $t = 1$ instead of currency compared to that of the equivalent case. The central bank only receives electronic money if the long term entrepreneur receives only electronic money and no currency. As additional amounts of currency are hoarded, the central bank holds more electronic money and less currency. Less money is deposited at the second bank since the hoarded currency is not deposited, but since all electronic money is deposited and this is the only amount of money the original bank borrows, L_1^B is the same and $L_1^B D_{1,2}^{ff}$ is the same. Since there are less deposits, $D_{1,2}$ increases but still satisfies $1 \leq D_{1,2} \leq \frac{D_2}{D_1}$. This is because the upper bound on $D_{1,2}$ is the same as the case of $P_1 \geq 1$, since the minimum deposits when $P_1 < 1$ with hoarding is the level of deposits when $P_1 \geq 1$, $(\lambda^w - \lambda)D_1$. So the change does not effect the results of the benchmark model.

At $t = 2$, the central bank redeems currency from the hoarders for its electronic money. Due to interest received, $D_{1,2} - 1$ on electronic money, the central bank is able to redeem more currency from hoarders than in the equivalent case. (The central bank never redeems additional currency at $t = 2$ from the long term entrepreneur because the central bank only receives electronic money and interest if the long term entrepreneur has no further currency at $t = 1$). Thus, the central bank is closer to satisfying its K^S liability. Since the hoarders have given up interest to the long term entrepreneur or the central bank (which spends its interest on currency taken out of circulation), the supply of dollars S_2 is lower at $t = 2$. Even if the central bank earns enough interest to redeem all currency and has extra to spend on goods at $t = 2$, it is interest that the hoarders do not receive, so S_2 is lower. Since $M_2^L \geq S_2$ in the benchmark model, a decrease in S_2 does not effect the results. Any increase of interest to the long term entrepreneur from an increase in $D_{1,2}$ is a decrease of interest to late consumers, so M_2^L and S_2 fall equally and $M_2^L \geq S_2$ continues to hold. The rest of the proof follows that of the benchmark model with the exception that the late hoarders consume less than by withdrawing at $t = 2$: $\frac{D_1}{P_2} < \frac{D_2}{P_2}$. Since the bank does not default, the hoarders strictly prefer to withdraw at $t = 2$, $\lambda^w = \lambda^p = \lambda$, which is a contradiction. Thus there is no hoarding and no run. ■

Proof of Proposition 4. Define $M_2^i \equiv K_2^i + K_1^i - Q_1^i P_1 D_{1,2}$, where

$$D_{1,2} \equiv \max\{\delta_2^A D_{1,2}^A,\ \delta_2^B D_{1,2}^B,\ \delta_2^C D_{1,2}^C\}.$$

L_1^i is the loan from bank i to bank A, and $D_{1,2}^{ff,i}$ is the return on L_1^i. Entrepreneur i's optimization problem at $t = 1$ is to maximize his total consumption of goods when there is no forced liquidation due to the default of bank i. This is similar to the benchmark model:

$$\max_{\beta_1^i, Q_2^i, \gamma_1^i} \quad [\overline{Q}_2^i(\beta_1^i, \gamma_1^i) - Q_2^i \mid \lambda^{w,j}, \lambda^{p,j} \; \forall \; j \in \{A, B, C\}] \tag{8.5a}$$

$$\text{s.t.} \quad (4.10\text{b}), (4.10\text{c})$$

$$\gamma_1^i \leq \alpha - \beta_0^i \tag{8.5b}$$

$$\beta_1^i \leq \beta_0^i + \gamma_1^i r \tag{8.5c}$$

$$(4.11\text{b}),$$

with the requirement that β_1^i, Q_2^i and γ_1^i are nonnegative. θ_1^i, θ_2^i, θ_3^i, θ_4^i and θ_5^i are the Lagrange multipliers associated with the constraints (4.10b), (4.10c), (8.5b), (8.5c) and (4.11b). The necessary Kuhn-Tucker conditions are (8.1a),

$$1 - \theta_1^i \frac{P_1}{P_2} D_{1,2} I_{[M_2^i \leq \overline{Q}_2^i P_2]} - \theta_1^i I_{[M_2^i > \overline{Q}_2^i P_2]} + \theta_2^i - \theta_4^i - \theta_5^i \leq 0 \qquad (= 0 \text{ if } \beta_1^i > 0) \tag{8.6a}$$

$$-R + \theta_1^i r \frac{P_1}{P_2} D_{1,2} I_{[M_2^i \leq \overline{Q}_2^i P_2]} + \theta_1^i R I_{[M_2^i > \overline{Q}_2^i P_2]} - \theta_2^i R - \theta_3^i + \theta_4^i r + \theta_5^i r \leq 0 \qquad (= 0 \text{ if } \gamma_1^i > 0), \tag{8.6b}$$

where $I_{[\cdot]}$ denotes the indicator function.

If $\frac{D_2}{P_2} \geq \frac{D_1}{P_1}$, late consumers choose to purchase goods at $t = 2$ so $\lambda^{w,B} = \lambda^{p,B} = \lambda^{w,C} = \lambda^{p,C} = \lambda - \epsilon$. If $\frac{D_2}{P_2} < \frac{D_1}{P_1}$, late consumers choose to withdraw and purchase early at $t = 1$ and $\lambda^{p,B} = \lambda^{p,C} = 1$. Intermediate cases of $\lambda < \lambda^{p,i} < 1$ for $i \in \{B, C\}$ require that $\frac{D_2}{P_2} = \frac{D_1}{P_1}$. Late consumers at bank B and C would not withdraw to redeposit unless the bank would default at $t = 2$ since, as in the benchmark model, they could not receive a greater return from another bank.

First I show that banks B and C each lending $L_1^B = L_1^C = \epsilon D_1$ to bank A at a return of $\frac{D_2}{D_1}$ is an equilibrium and corresponds to the benchmark first best outcome. Consider $\lambda^{w,i} = \lambda^{p,i} = \lambda$, $D_1 = C_1^*$, $D_2 = C_2^*$, $Q_1^i = \lambda D_1$, $Q_2^i = (1 - \lambda) D_2$, $\beta_1^i = \gamma_1^i = 0$ and $D_{1,2}^i = 1 \; \forall \; i \in \{A, B, C\}$. From the definition of prices, $P_1 = P_2 = 1$. There are no bank defaults, and this solution satisfies the entrepreneurs' constraints from (8.5) and first-order conditions, (8.1a) and (8.6), and is a maximum for the objective function in (8.5a), and the late consumers' problems, similar to the benchmark model, and so is an equilibrium. $C_1^i = \frac{D_1}{P_1} = C_1^*$ and $C_2^i = \frac{D_2}{P_2} = C_2^*$ is a first best outcome.

Next, I show that banks B and C both not lending to A is an equilibrium. From the definition of prices,

$$P_1 = \frac{(\delta_1^A \lambda^{p,A} + \lambda^{p,B} + \lambda^{p,C}) D_1}{Q_1^A + Q_1^B + Q_1^C} \tag{8.7}$$

$$P_2 = \frac{[\delta_2^A (1-\lambda^{w,A}) + (1-\lambda^{w,B}) + (1-\lambda^{w,C})] D_2}{Q_2^A + Q_2^B + Q_2^C}. \tag{8.8}$$

Let $\widetilde{\gamma}_1^A = \frac{\gamma_1^A}{\alpha}$ be the fraction of α goods originally invested that are liquidated by entrepreneur A at $t = 1$. The amount of money bank A pays to depositors at $t = 1$ must equal the money it receives from entrepreneur A plus loans from banks B and C:

$$\delta_1^A \lambda^{w,A} D_1 = P_1[\lambda D_1 + (1-\lambda)\, D_2 \frac{r}{R} \widetilde{\gamma}_1^A] + L_1^B + L_1^C. \tag{8.9}$$

Similarly, the money bank A pays to depositors and repays for interbank loans to banks B and C at $t = 2$ must equal the money it receives from entrepreneur A:

$$\delta_2^A (1-\lambda^{w,A}) D_2 + (L_1^B D_{1,2}^{ff,B} + L_1^C D_{1,2}^{ff,C}) = P_2 Q_2^A. \tag{8.10}$$

Consider the solution $\lambda^{p,i} = \lambda^{w,i}$ and $D_{1,2}^i = 1$ for $i \in \{A, B, C\}$; $Q_1^i = \lambda D_1$, $Q_2^i = (1-\lambda)\, D_2$ and $\lambda^{p,i} < \lambda$ for $i \in \{B, C\}$; and $Q_1^A \geq \lambda D_1$ and $L_1^B = L_1^C = 0$. Substituting into (8.7) and solving for P_1 gives

$$P_1 = \frac{(\lambda^{p,B} + \lambda^{p,C}) D_1}{2\lambda D_1}. \tag{8.11}$$

Substituting into (8.8) and solving for P_2 gives

$$P_2 = \frac{(1-\lambda^{w,B} + 1 - \lambda^{w,C}) D_2}{2\,(1-\lambda)\, D_2}. \tag{8.12}$$

This implies $P_1 < 1$ and $P_2 > 1$, banks B and C do not default, and the conjectured solution satisfies the entrepreneurs' constraints from (8.5), and first-order conditions, (8.1a) and (8.6), and is a maximum for the objective function in (8.5a),and satisfies the late consumers' problems. Thus the outcome of banks B and C not lending to bank A is an equilibrium. Since $P_1 < 1$ and $\lambda^{w,A} > \lambda$, (8.9) implies that $\widetilde{\gamma}_1^A > 0$. Since $P_1 < 1$ and $\frac{D_2}{D_1} < R$, (8.9) and (8.10) together imply $\delta_2^A < 1$. $C_2^i = \frac{\delta_2^i D_2}{P_2} < C_2^*$ implies suboptimal consumption sharing for all consumers. ■

Proof of Proposition 5. Late consumers of bank A run the bank if $\frac{\delta_2^A D_2}{P_2} < \frac{\delta_1^A D_1}{P_1}$. If $\delta_1^A < 1$, the bank defaults at $t = 1$ and must liquidate all investments. This implies $\delta_2^A = 0$, so the condition holds. Consider $\delta_1^A = 1$. Solving (8.9) for $\widetilde{\gamma}_1^A$ and (8.10) for δ_2^A and substituting both along with P_1 from (8.11) and P_2 from (8.12) into the condition for the bank A run, after rearranging, gives $\epsilon > \widetilde{\epsilon}^{w,A}$, where

$$\widetilde{\epsilon}^{w,A} \equiv \frac{r\lambda\left(1-\lambda\right)\left(D_2-D_1\right)-\lambda D_1(1-r)(\lambda^{w,A}-\lambda^{A})+\left[\lambda R D_1+\left(1-\lambda\right) r D_2\right](\lambda^{w,B}-\lambda^{B})}{\lambda(3R-2r)D_1+r\left(1-\lambda\right)D_2}$$

If $\frac{\delta_2^A D_2}{P_2} < \frac{D_1}{P_1} < \frac{D_2}{P_2}$, late consumers of banks B and C do not run while late consumers of banks A do run. This implies the bank A run condition holds when $\lambda^{w,B} = \lambda^B$. Since $\lambda^{w,A} \geq \lambda^A$, $\widetilde{\epsilon}^{w,A} \leq \widetilde{\epsilon}^{A} = \frac{r\lambda(1-\lambda)(D_2-D_1)}{\lambda(3R-2r)D_1+r(1-\lambda)D_2}$. Thus, $\epsilon > \widetilde{\epsilon}^{A}$ implies $\epsilon > \widetilde{\epsilon}^{w,A}$ and the condition holds. Since $\widetilde{\epsilon}^{w,A}$ is decreasing in $\lambda^{w,A}$, the condition holds for any size of bank run $\lambda^{w,A}$. This implies that there is a full run and $\lambda^{w,A} = 1$. ■

Proof of Proposition 6. Since the condition for runs of banks B and C by late consumers is the same, $\lambda^{p,B} = \lambda^{p,C}$. Since there is no redepositing as shown in Proposition 5, $\lambda^{w,B} = \lambda^{p,B}$. The condition for runs by late consumers of banks B and C is $\frac{D_2}{P_2} < \frac{D_1}{P_1}$. Substituting for prices from (8.11) and (8.12) and rearranging gives $\epsilon > \widetilde{\epsilon}^{B,C} \equiv \frac{\lambda(1-\lambda)(D_2-D_1)}{\lambda D_1+(1-\lambda)D_2}$.

Next I show the run on banks B and C is a partial run. Suppose not: $\lambda^{w,B} = 1$. A run on banks B and C to purchase goods implies $\frac{D_1}{P_1} \geq \frac{D_2}{P_2}$. Substituting for $P_1 = \frac{\lambda^{w,B}}{\lambda}$, $P_2 = \frac{(1-\lambda^{w,B})}{(1-\lambda)}$ and $\lambda^{w,B} = 1$, and rearranging implies $(1-\lambda)\,D_2 \leq 0$, a contradiction to the assumption $\lambda < 1$. Thus $\lambda^{w,B} < 1$. Moreover, $\widetilde{\epsilon}^{B,C} > \widetilde{\epsilon}^{A}$. Suppose not: $\widetilde{\epsilon}^{A} \geq \widetilde{\epsilon}^{B,C}$. Simplifying, this implies $r \geq R$, a contradiction. Thus $\epsilon > \widetilde{\epsilon}^{B,C}$ implies $\epsilon > \widetilde{\epsilon}^{A}$ and bank A is fully run. ■

Proof of Proposition 7. (8.9) is replaced by

$$\delta_1^A \lambda^{w,A} D_1 = P_1[\lambda D_1 + (1-\lambda)\,D_2 \frac{r}{R}\widetilde{\gamma}_1^A] + L_1^B + L_1^C + L_1^{CB}$$

and (8.10) is replaced by

$$\delta_2^A (1-\lambda^{w,A}) D_2 + (L_1^B D_{1,2}^{ff,B} + L_1^C D_{1,2}^{ff,C}) + L_1^{CB}\left(\frac{D_2}{D_1}\right) = P_2 Q_2^A,$$

where the central bank loan is denoted as L_1^{CB}. The guaranteed central bank loan implies $L_1^B + L_1^C + L_1^{CB} = 2\epsilon D_1$. This implies prices are given by

$$\begin{aligned} P_1 &= \frac{[2\lambda + (\lambda^{w,B}-\lambda^B) + (\lambda^{w,C}-\lambda^C)]D_1}{Q_1^B + Q_1^C} \\ P_2 &= \frac{[2\,(1-\lambda) + (1-\lambda^{w,B}) + (1-\lambda^{w,C})]D_2}{Q_2^B + Q_2^C}. \end{aligned}$$

The entrepreneurs' constraints from (8.5), and first-order conditions, (8.1a) and (8.6), and the late consumers' problem implies a unique solution of no runs, $Q_1^i = \lambda D_1$ and $Q_2^i = (1-\lambda)\,D_2$, if bank B and C do not default at $t = 2$, following the proof from the benchmark model, so

$P_1 = P_2 = 1$. The budget constraint for bank $i \in \{B, C\}$ is

$$\begin{aligned}
\delta_1^i \lambda^{w,i} D_1 &= \lambda D_1 - L_1^i \\
\delta_2^i (1 - \lambda^{w,i}) D_2 &= (1 - \lambda) D_2 + L_1^i D_{1,2}^{ff,i},
\end{aligned}$$

which implies that bank $i \in \{B, C\}$ chooses $L_1^i = \epsilon D_1$ and does not default. Thus the solution is a unique equilibrium. ■

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