Technical Appendix for "Stabilizing Expectations Under Monetary and Fiscal Policy Coordination"

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Abstract

This technical appendix provides i) some calculations underlying the model used in Eusepi and Preston (2008); and ii) some additional results that both clarify our findings relative to earlier learning analyses on this topic and elucidate further the role of some assumptions.

Contents

1 Model Derivation

The following describes only the derivation of the aggregate demand equation. The derivation of the generalized Phillips curve can be found in Preston (2005b). The household's optimality conditions imply:

$$E_0^i \sum_{t=0}^{\infty} \beta^t \frac{U_c\left(C_t^i+g\right)}{U\left(C_0^i+g\right)} C_t^i = \frac{B_0^i}{P_0} + E_0^i \sum_{t=0}^{\infty} \beta^t \frac{U_c\left(C_t^i+g\right)}{U_c\left(C_0^i+g\right)} \left[Y_t - \frac{T_t}{P_t}\right]$$

which can be rewritten as

$$b_0^i \pi_0^{-1} = E_0^i \sum_{t=0}^{\infty} \beta^t \frac{U_c\left(C_t^i+g\right)}{U\left(C_0^i+g\right)} \left[C_t^i - Y_t + \tau_t\right] \tag{1}$$

where $\tau_t = T_t/P_t$ and $b_t^i = B_t^i/P_t$. In steady $\bar{s} = (1-\beta)\,\bar{b}$ where $s_t = T_t/P_t - g$ defines the structural surplus and market clearing implies $\bar{Y} = \bar{C} + g$.

Approximating (1) provides

$$\begin{aligned}
\bar{b}\left(\hat{b}_0^i - \hat{\pi}_0\right) &= \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [\bar{C}\hat{C}_t^i - \bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t] + \bar{s}\hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left[\frac{U_{cc}}{U_c}\bar{C}\hat{C}_t^i - \frac{U_{cc}}{U_c}\bar{C}\hat{C}_0^i\right] \\
&= \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [\left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right)\hat{C}_t^i + \bar{s}\tilde{\sigma}^{-1}\hat{C}_0^i\,(1-\beta)^{-1} + \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [-\bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t]
\end{aligned}$$

where $\tilde{\sigma} = -U_c/\left(U_{cc}\bar{C}\right)$.

The consumption Euler equation satisfies the log-linear approximation

$$\hat{C}_t^i = \hat{E}_t^i \hat{C}_{t+1}^i - \tilde{\sigma}\left(\hat{\imath}_t - \hat{E}_t^i \hat{\pi}_{t+1}\right).$$

Solving recursively backwards and taking expectations at time zero provides

$$\hat{E}_0^i \hat{C}_t^i = \hat{C}_0^i + \tilde{\sigma}\hat{E}_0^i \sum_{s=0}^{t-1} \left(\hat{\imath}_s - \hat{\pi}_{s+1}\right).$$

This determines the infinite sum

$$\begin{aligned}
\hat{E}_0^i \sum_{t=1}^{\infty} \beta^t \hat{C}_t^i &= \frac{\beta \hat{C}_0^i}{(1-\beta)} + \tilde{\sigma}\hat{E}_0^i \sum_{t=1}^{\infty} \beta^t \sum_{s=0}^{t-1} \left(i_s - \pi_{s+1}\right) \\
&= \frac{\beta \hat{C}_0^i}{(1-\beta)} + \frac{\tilde{\sigma}\beta}{(1-\beta)}\hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left(i_t - \pi_{t+1}\right).
\end{aligned}$$

Substituting into the intertemporal budget constraint

$$\begin{aligned}
\bar{b}\left(\hat{b}_0^i - \hat{\pi}_0\right) &= \hat{E}_0^i \sum_{t=1}^{\infty} \beta^t \left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right) \hat{C}_t^i + \left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right) \hat{C}_0^i + \bar{s}\tilde{\sigma}^{-1} \frac{\hat{C}_0^i}{(1-\beta)} + \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [-\bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t] \\
&= \frac{\bar{C}\hat{C}_0}{(1-\beta)} + \left(\bar{C} - \bar{s}\tilde{\sigma}^{-1}\right) \frac{\tilde{\sigma}\beta}{(1-\beta)} \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left(i_t - \pi_{t+1}\right) + \hat{E}_0^i \sum_{t=0}^{\infty} \beta^t [-\bar{Y}\hat{Y}_t + \bar{\tau}\hat{\tau}_t]
\end{aligned}$$

Divide through by $\bar{Y}\ (1-\beta)^{-1}$ and rearranging gives the optimal consumption rule

$$\hat{C}_0^i = s_C^{-1}\delta\left(\hat{b}_0 - \hat{\pi}_0\right) + s_C^{-1}\hat{E}_0^i \sum_{t=0}^{\infty} \beta^t \left[(1-\beta)\left(\hat{Y}_t - \delta\hat{s}_t\right) - (\sigma - \delta)\beta\left(\hat{\imath}_t - \hat{\pi}_{t+1}\right)\right]$$

where $s_C = \bar{C}/\bar{Y}$, $\delta = \bar{s}/\bar{Y}$, $\bar{s}\hat{s}_t = \bar{\tau}\hat{\tau}$, $\sigma = s_c\tilde{\sigma}$ and using $\bar{s} = (1-\beta)\,\bar{b}$.

Finally, note that market clearing implies the log-linear approximations

$$\hat{Y}_t = s_C \int_0^1 \hat{C}_t^i di \text{ and } \hat{b}_t = \int_0^1 \hat{b}_t^i di.$$

Hence the aggregate demand equation is

$$\hat{Y}_0 = \delta\left(\hat{b}_0 - \hat{\pi}_0\right) + \hat{E}_0 \sum_{t=0}^{\infty} \beta^t \left[(1-\beta)\left(\hat{Y}_t - \delta\hat{s}_t\right) - (\sigma - \delta)\beta\left(\hat{\imath}_t - \hat{\pi}_{t+1}\right)\right]$$

where $\hat{E}_t = \int_0^1 \hat{E}_t^i di$ defines average beliefs of households.

Define the output gap as $Y_t - Y_t^n$ where the latter is the natural rate of output under rational expectations permits

$$x_t = \delta\beta^{-1}\left(\hat{b}_t - \hat{\pi}_t\right) + \hat{E}_t \sum_{T=t}^{\infty} \beta^t \left[(1-\beta)\left(\hat{x}_{T+1} - \delta\beta^{-1}\hat{s}_T\right) - (\sigma - \delta)\left(\hat{\imath}_t - \hat{\pi}_{t+1}\right) + r_T\right]$$

where

$$r_t = Y_{t+1}^n - Y_t^n.$$

Equation (10) of the paper follows when $\tilde{\sigma} = 1$ and assuming $g = 0$, without loss of generality, so that $\tilde{\sigma} = \sigma = 1$.

2 Alternative Models of Learning Dynamics

Many recent papers have proposed analyses of learning dynamics in the context of models where agents solve infinite-horizon decision problems, but without requiring that agents make forecasts more than one period into the future. In these papers, agents' decisions depend only on forecasts of future variables that appear in Euler equations used to characterize rational expectations equilibrium. Important contributions include Bullard and Mitra (2002) and Evans and Honkapohja (2003). Of most relevance to the present study is Evans and Honkapohja (2007) which similarly studies the interaction of monetary and fiscal policy, but in a model of learning dynamics in which only one-period-ahead expectations matter to expenditure and pricing plans of households and firms. The following section replicates part of their analysis in the context of the model developed here, and contrasts the resulting findings with those of sections 5 and 6.

Since the optimal decision rules for households and firms presented in section 2 are valid under arbitrary assumptions on expectations formation, they are satisfied under the rational expectations assumption. Application of this assumption implies the law of iterated expectations to hold for the aggregate expectations operator and permits simplification of relations (10) and (11) in the paper to the following aggregate Euler equation and Phillips curve:[1]

$$\begin{aligned}\hat{x}_t &= E_t\hat{x}_{t+1} - (\hat{\imath}_t - E_t\hat{\pi}_{t+1} - r_t) \\ \hat{\pi}_t &= \kappa\hat{x}_t + \beta E_t\hat{\pi}_{t+1}.\end{aligned}$$

Under learning dynamics, with only one-period-ahead expectations, it is assumed that aggregate demand and supply conditions are determined by

$$\hat{x}_t = \hat{E}_t\hat{x}_{t+1} - \left(\hat{\imath}_t - \hat{E}_t\hat{\pi}_{t+1} - r_t\right) \tag{2}$$

$$\hat{\pi}_t = \kappa\hat{x}_t + \beta\hat{E}_t\hat{\pi}_{t+1}. \tag{3}$$

Identical assumptions are made on monetary and fiscal policy provide the remaining model

[1] See Preston (2005a, 2005b) for a detailed discussion.

equations

$$\hat{\imath}_t = \phi_\pi E^{cb}_{t-1}\hat{\pi}_t \tag{4}$$

$$\hat{s}_t = \phi_\tau \hat{b}_t \tag{5}$$

and

$$\hat{b}_{t+1} = \beta^{-1}\left(\hat{b}_t - \hat{\pi}_t - (1-\beta)\,\hat{s}_t\right) + \hat{\imath}_t \tag{6}$$

The model is closed with a description of beliefs. As nominal interest rates and taxes need not be forecast, an agent's vector autoregression model is estimated on the restricted state vector

$$X_t = \begin{bmatrix} \hat{x}_t \\ \hat{\pi}_t \\ \hat{b}_{t+1} \end{bmatrix}.$$

Two points should be underscored. First, the assumption that only one-period-ahead forecasts matter, implies that households and firms do not take account of transversality conditions in making their spending and pricing plans. Decisions are not optimal given maintained beliefs. That households fail to make decisions that satisfy their intertemporal budget constraint might be thought to have implications in the present context as the fiscal theory of the price level is a theory grounded on shifting evaluations of various variables related precisely by this constraint. Furthermore, by ignoring the implications of the intertemporal budget constraint, fiscal policy has no direct impact on spending and pricing decisions. Neither forecasts of future taxes nor the average indebtedness of the macroeconomy matter for aggregate dynamics. Second, and related, is that because households do not need to forecast future nominal interest rates or taxes there is no uncertainty about the policy rules adopted by the monetary and fiscal authority — there is no regime uncertainty and no role for communication of the joint policy strategy. It seems worth exploring the consequences of these alternative modeling assumptions, and learning whether they elucidate earlier results.

In the model given by relations (2), (3), (4), (5) and (6) the following stability results obtain.

Proposition 1 *For $0 < \alpha < 1$, stabilization policy ensures expectational stability if and only if the following conditions are satisfied: either*

1. *Monetary policy is active and fiscal policy is locally Ricardian such that*

$$\phi_\pi > 1 \text{ and } 1 < \phi_\tau < \frac{1+\beta}{1-\beta} \text{ ; or}$$

2. *Monetary policy is passive and fiscal policy is non-Ricardian such that*

$$0 \leq \phi_\pi < 1 \text{ and either } 0 \leq \phi_\tau < 1 \text{ or } \phi_\tau > \frac{1+\beta}{1-\beta}.$$

This generalizes the Evans and Honkapohja (2006) analysis to a model with nominal rigidities.[2] When only one-period-ahead expectations matter, the Leeper conditions are sufficient to rule out expectations-driven instability. In contrast, in a model of optimal decisions, these conditions obtain only if there is no regime uncertainty — i.e. the policy rules are credibly communicated to households and firms — and either agents believe the government accounts to be intertemporally solvent or the fiscal authority chooses policy so that the steady state debt-to-output ratio is zero. If neither of these conditions is met, the analysis of this paper suggests a smaller menu of policies is consistent with expectations stabilization. Furthermore, economies with non-zero debt-to-output ratios experience rather different dynamics in response to disturbances — recall the impulse response functions of the previous section.

3 Additional Propositions

Two further results are presented in Eusepi and Preston (2008) without proof. The first regards a special case of proposition two for a general assumption on the degree of nominal rigidity. For a specific configuration of policy, analytic results are available. The second regards the central bank's imperfect knowledge about the state of the economy. When the central bank can observe current inflation, the Leeper conditions are necessary and sufficient for stability under learning dynamics.

3.1 Arbitrary Nominal Rigidity

Proposition 2 *Assume agents face uncertainty about the monetary and fiscal regimes. If $\phi_\pi = 0$ and $\tau = 0$, then the rational expectations equilibrium is stable for all parameter values.*

[2] The proof is available on request.

Proof of Proposition 2

Assuming $\phi_\pi = 0$ implies that $\lambda_2 = \alpha$. In fact,

$$\begin{aligned}\lambda_2 &= \frac{1}{2\beta}\left[1+\beta+\kappa(\alpha)-\sqrt{(1+\beta+\kappa(\alpha))^2-4\beta}\right] \\ &= \frac{1}{\beta}\left(\frac{1}{2}\beta+\frac{1}{2\alpha}\left(\alpha^2\beta-\alpha\beta-\alpha+1\right)-\frac{1}{2\alpha}\left(1-\alpha^2\beta\right)+\frac{1}{2}\right) \\ &= \alpha\end{aligned}$$

The proof proceeds in the same steps as for Proposition 2. The expressions below are obtained from the file[3] *fiscal_ delay_ benchmark.m.* The trace of the $\tilde{A}$ matrix describing the dynamics of the intercept becomes

$$tr(\tilde{A})+1=-\frac{\left[\beta^2\alpha^2-\alpha\left(\beta^2+2\beta\right)+2\right]}{1-\alpha\beta}<0$$

while the determinant of the matrix $\tilde{A}$ is equal to -1 for every parameter value. Consider the B matrix (dynamics of the b coefficients): the traces is

$$tr(\tilde{B})+1=\frac{\alpha^4\beta^3-\alpha^4\beta^2-\alpha^3\beta^3-\alpha^3\beta^2+2\alpha^2\beta+2\alpha\beta-2}{(1-\alpha\beta)(1-\alpha^2\beta)}=\frac{G(\alpha)}{(1-\alpha\beta)(1-\alpha^2\beta)}.$$

Consider the numerator $G(\alpha)$. It is straightforward to show that

$$G(0)=-2 \text{ and } G(1)=-2\beta^3-2\beta^2+2\beta+2\beta-2<0.$$

Finally,

$$\begin{aligned}G'(\alpha) &= -4\alpha^3\beta^2(1-\beta)-4\alpha^2\beta^2+\alpha^2\beta^2+4\alpha\beta+2\beta-3\alpha^2\beta^3= \\ &\quad \left[-4\alpha^3\beta^2(1-\beta)+4\alpha\beta(1-\alpha\beta)\right]+\alpha^2\beta^2+2\beta-3\alpha^2\beta^3 \\ &> \left[-4\alpha^3\beta^2(1-\beta)+4\alpha\beta(1-\alpha\beta)\right]+\alpha^2\beta^2+2\alpha^2\beta^2-3\alpha^2\beta^3 \\ &= \left[-4\alpha^3\beta^2(1-\beta)+4\alpha\beta(1-\alpha\beta)\right]+3\alpha^2\beta^2-3\alpha^2\beta^3>0\end{aligned}$$

showing that the trace is always negative. The determinant is always equal to -1.

[3] In order to generate the result, select the option "peg=1".

3.2 Resolving Central Bank Uncertainty

Proposition 3 *Assume the central bank can perfectly observe current inflation. Then the stability conditions under learning coincide with the conditions for local determinacy.*

Proof of Proposition 3

The Proposition follows the same steps as Proposition 2. The expressions below are obtained by using the file *fiscal_ current_ analytical.m*. Under perfect information on the part of the central bank, *both* matrices A and B can be further reduced by exploiting an extra restriction on their parameters. For matrix A,

$$A_{3,j} = \phi_\pi A_{2,j} \;\; \text{for } j = 1...5$$

and the same for matrix B. The reduced matrices $\tilde{A}$ and $\tilde{B}$ are in this case only two dimensional. Real negative eigenvalues require positive determinant and negative trace.

Ricardian regime. As $\alpha \to 0$, trace and determinant of matrix $\tilde{A}$ converge to

$$\lim_{\alpha\to 0} tr\left(\tilde{A}\right) = -\frac{2-\beta}{1-\beta} + \frac{1}{(1-\beta)\,\phi_\pi} < 0$$

and

$$\lim_{\alpha\to 0} \det\left(\tilde{A}\right) = \frac{\phi_\pi - 1}{\phi_\pi\,(1-\beta)} > 0$$

provided $\phi_\pi > 0$. For the matrix $\tilde{B}$, the trace is

$$tr(\tilde{B}) = \frac{-\phi_\tau\,(1-\beta)\,(\beta\phi_\pi + 1) - \beta\phi_\pi + 1}{\phi_\pi\beta(1-\beta)\phi_\tau}$$

which can be verified t be negative if $\phi_\tau > 1$ (as required in the Ricardian regime) and $\phi_\pi > 1$. Finally, the determinant is

$$\det(\tilde{B}) = \frac{-\phi_\tau\,(1-\beta) + 1 - \beta\phi_\pi}{\phi_\pi\beta(1-\beta)\phi_\tau}$$

is negative if $\phi_\tau > 1$ (as required in the Ricardian regime) and $\phi_\pi > 1$.

Non-Ricardian **regime**.Solving for the determinant of $\tilde{A}$ we get

$$\lim_{\alpha\to 0} \det(\tilde{A}) = \frac{1-\phi_\tau}{1-(1-\beta)\,\phi_\tau} > 0$$

Values of ϕ_τ consistent with a non-Ricardian fiscal rule satisfy

$$-1 < H(\phi_\tau) = \frac{\beta}{1-(1-\beta)\phi_\tau} < 1.$$

Multiplying the determinant by β (which leaves its sign unchanged) we get that the condition to be satisfied is

$$\frac{\beta(1-\phi_\tau)}{(1-\phi_\tau+\phi_\tau\beta)} > 0. \tag{7}$$

for $\phi_\tau < 1$ and $\phi_\tau > \frac{1+\beta}{1-\beta}$. Consider the trace. We obtain

$$\begin{aligned}
\lim_{\alpha\to 0} tr(\tilde{A}) &= -\frac{(\phi_\tau\beta+2-2\phi_\tau)}{(1-\phi_\tau+\phi_\tau\beta)} \\
&= -\left[1+\frac{1-\phi_\tau}{(1-\phi_\tau+\phi_\tau\beta)}\right] < 0.
\end{aligned}$$

Concerning the matrix $\tilde{B}$, the trace is

$$\lim_{\alpha\to 0} tr(\tilde{B}) = G^T(\phi_\pi) - \frac{(\beta^2\phi_\pi\phi_\tau-\beta\phi_\pi\phi_\tau+2\beta\phi_\pi-2\phi_\tau\beta+2\phi_\tau-2)}{(\beta\phi_\pi-1)(1-\phi_\tau+\phi_\tau\beta)}.$$

In the case $\phi_\pi = 0$ we get

$$G^T(0) = -\frac{2(1-\phi_\tau+\phi_\tau\beta)}{(1-\beta)(1-\phi_\tau+\phi_\tau\beta)} = -\frac{2}{(1-\beta)} < 0$$

and for $\phi_\pi = 1$,

$$G^T(1) = -\frac{(\beta^2\phi_\tau-\beta\phi_\tau+2\beta-2\phi_\tau\beta+2\phi_\tau-2)}{(\beta-1)(1-\phi_\tau+\phi_\tau\beta)} = -\left[1+\frac{1-\phi_\tau}{(1-\phi_\tau+\phi_\tau\beta)}\right] < 0$$

Last,

$$\begin{aligned}
G'^T(\phi_\pi) &= \frac{\beta\phi_\pi(\beta-1)}{(1-\phi_\tau+\phi_\tau\beta)^2(\beta\phi_\pi-1)} \\
&= \frac{\beta\phi_\pi(1-\beta)}{(1-\phi_\tau+\phi_\tau\beta)^2(1-\phi_\pi\beta)} \geq 0 \text{ for every } \phi_\pi \in [0,1]
\end{aligned}$$

which implies that the trace is negative for every value of ϕ_π and ϕ_τ consistent with the determinate and stationary REE. The determinant,

$$\lim_{\alpha\to 0} \det(\tilde{B}) = \frac{1-\phi_\pi\beta-(1-\beta)\,\phi_\tau}{(1-\phi_\pi\beta)(1-\phi_\tau\,(1-\beta))} > 0$$

for $\phi_\tau < 1$ and $\phi_\tau > \frac{1+\beta}{1-\beta}$.

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