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Estimating the Term Structure of
Interest Rates with Simple and Complex
Functional Forms: Nelson & Siegel vs.
Longstaff & Schwartz

*Magnus Dahlquist and
Lars E. O. Svensson*

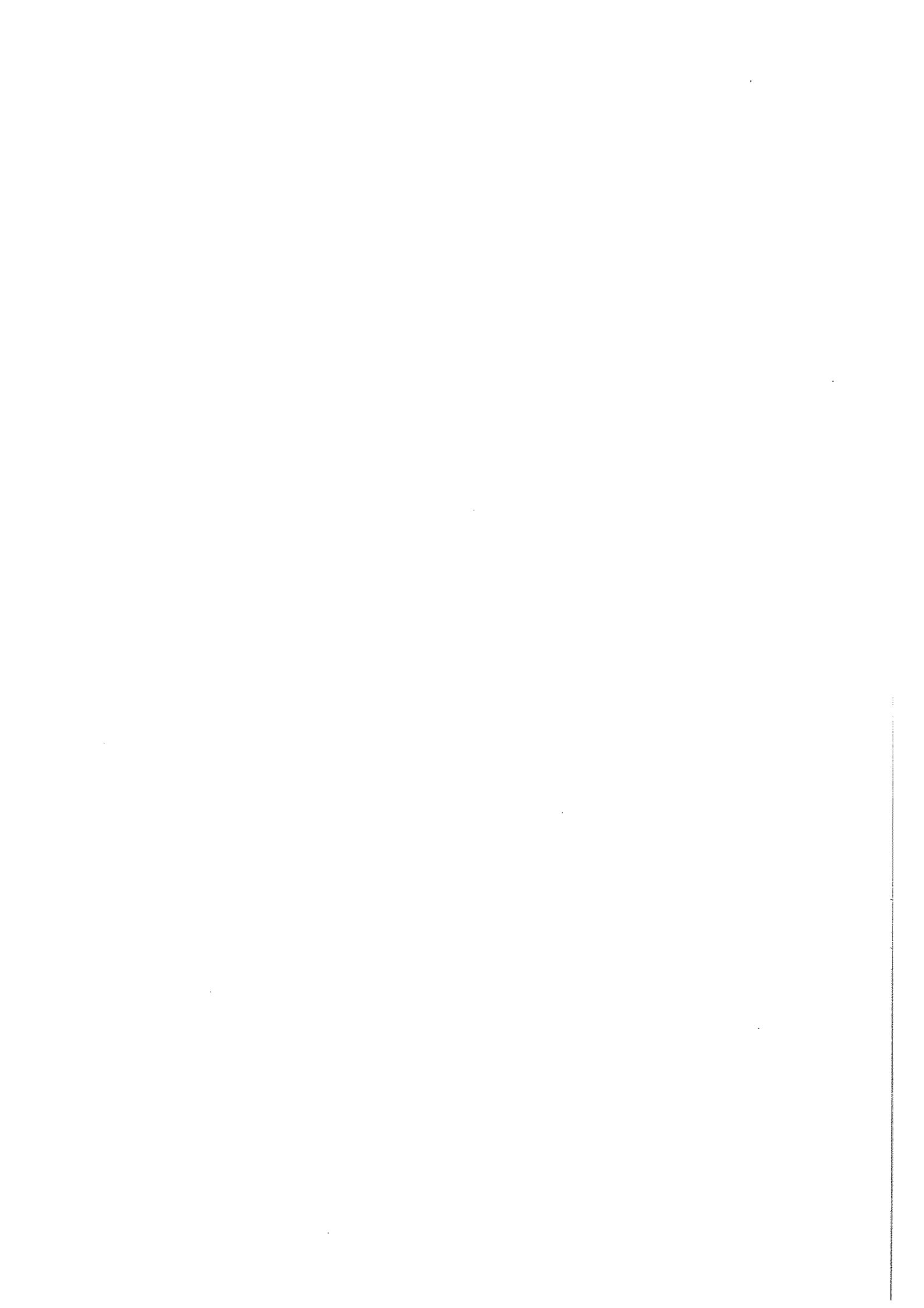
Abstract

The paper compares estimation of spot (zero-coupon) interest rates and implicit forward interest rates from Swedish Treasury bill rates and Government coupon bond yields with two functional forms for the discount function, the simple form of Nelson & Siegel (NS) and the complex form of Longstaff & Schwartz (LS). NS is much easier to use and has much better convergence properties, whereas LS is more flexible. For the data used, estimates with NS and LS are close, with only marginally better fit for LS. The fit of NS seems satisfactory for monetary policy purposes.

Key words: Discount function, forward rates, spot rates.

JEL classification numbers: E43, E52, G12.

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1 Introduction

The term structure of interest rates, that is, how interest rates depend on the time to maturity, receives considerable attention in both financial and economic analysis. Estimated spot interest rates (zero-coupon rates) for different maturities and associated implied forward interest rates are since long standard tools for financial analysis in the financial markets, for instance in the pricing of financial instruments. The term structure of interest rates, in the form of the yield curve (that is, the yield to maturity on coupon bonds as a function of their time to maturity) and the yield spread between long and short interest rates (that is, the slope of the yield curve), is also a traditional indicator for monetary policy. Long interest rates are usually considered to vary with long-run inflation expectations, and the spread between long and short interest rates is sometimes interpreted as indicating how expansionary or contractionary current monetary policy is.

The recent move to flexible exchange rates in Europe is likely to increase the role of indicators in monetary policy. A fixed exchange rate can be seen as a well-defined intermediate target for monetary policy. The loss of an intermediate target and the difficulties in finding a new intermediate target make it likely that monetary policy will be more dependant upon the use of indicators. As a complement or even alternative to the standard yield curve and the yield spread, implied forward interest rates have recently begun to be used as one of the monetary policy indicators by several central banks, for instance Bank of England, Federal Reserve Board, and Sveriges Riksbank. Implied forward interest rates are interest rates on loans and investments that start on a future date. They can be derived from the yield curve. Forward rates present the information in the term structure of interest rates in a way that is more easily interpreted for monetary policy purposes, and can under appropriate assumptions be used to infer market expectations of future interest rates, inflation rates and currency depreciation rates more directly than the standard yield curve (see Bank of England (1993) and Svensson (1993b,c)).

The standard yield curve of yields to maturity on Treasury bills and Government bonds plotted against the time to maturity is unfortunately not an unambiguous representation of the term structure of interest rates, even if the bonds have no default risk and are fully liquid. The reason is that almost all bonds are coupon bonds, and yields to maturity on coupon bonds depend on the coupon, the "coupon effect." Bonds maturing at the same date with different coupons will therefore show different yields to maturity. The term structure therefore need to be expressed in some standardized way in order to be unambiguous.

Two standardized ways to express the term structure occur in the literature, namely to report a par yield curve consisting of yields to maturity on par bonds (bonds that trade at par and have coupons that equal the yield to maturity) or to

report a spot rate curve consisting of yields to maturity on zero-coupon bonds. In practice there are few bonds trade at par, and there are few or no zero-coupon bonds available beyond 12 months maturity. Either way to express the term structure then requires estimation of the term structure from yields to maturity on non-par coupon bonds. Even if there were a fair number of par bonds or zero-coupon bonds available, it would still be advantageous to also use the information in the non-par coupon bond yields.¹

The most common way to express the term structure of interest rates these days is in terms of spot rates that are estimated from available coupon bonds for longer maturities and zero-coupon bonds (Treasury bills) for shorter maturities. Implicit forward rates are then calculated from these spot rates. A number of different estimation methods are available.

McCulloch (1971, 1975) introduced the now standard method to estimate a continuous discount function (the zero-coupon bond prices as a function of the time to maturity) from coupon bond prices. The method consists of computing model prices of the bonds by valuing the coupon payments and the principal with the discount function, and then estimate the parameters of the discount function by fitting model prices to actual bond prices by minimizing the sum of the squared errors between model and actual prices. The spot and forward rates can then be computed from the discount function. McCulloch (1975) used a cubic spline as the functional form for the discount function, which allows the estimation to be formulated as a linear regression. As discussed by Shea (1984), the cubic spline has the disadvantage that estimates of forward rates may be unstable. Especially for the longest maturity in the sample they may fluctuate between large positive and negative values, and they may even be negative. The estimates also depend on the location of the knot points between the different segments of the cubic spline.

We shall follow McCulloch in estimating a discount function by fitting model prices to observed bond prices. We shall deviate from McCulloch in using two other functional forms than the cubic spline, one very simple form suggested by Nelson and Siegel (1987) (NS), and one very complex form derived in a theoretical model by Longstaff and Schwartz (1992) (LS). These functional forms give more stable estimates of forward rates than cubic splines. The forms are not nested.

The purpose of the paper is hence to compare the performance of the NS and LS functional forms in estimating spot and forward rates on Swedish term

¹ The term structure can also be estimated from the interbank interest-swap market, since the yields quoted there can be interpreted as yields on par bonds. The yields may include a default risk, though.

structure data in order to judge which method is most appropriate for monetary policy analysis.²

Thus monetary policy analysis rather than financial analysis is in focus, for instance, the use of forward rates as indicators rather than to price financial instruments for arbitrage decisions. This has some consequence for the criteria according to which the performance of the two functional forms is evaluated. *First*, for monetary policy purposes somewhat less precision is required than for financial analysis. Yield errors of a few tens of basis points are acceptable for monetary policy analysis, but hardly acceptable for arbitrage decisions. *Second*, from an economics point of view, zero-coupon prices can be interpreted as intertemporal marginal rates of substitution. It seems reasonable to postulate that marginal rates of substitution are rather smooth, in which case it follows that estimates spot and forward rates should be rather smooth. Smoothness reduces precision, but as mentioned the demand for precision is less for monetary policy analysis than for financial analysis. Increased demand for precision in financial analysis tends to result in jagged spot rates and volatile forward rates. *Third*, since forward rates can be interpreted as indicating expectations of future interest rates, which in turn depend on expectations of future real interest rates and future inflation rates, it seems reasonable to restrict forward rates for settlements very far into the future to be constant. This because it seems unlikely that market agents have information that allow them to have different expectations for, say, 25 and 30 years into the future. *Fourth*, the demands on robustness of estimates is probably higher for monetary policy analysis than for financial analysis. The estimates in policy analysis should allow comparisons over time and across countries, with different sets of bonds and Treasury bills, and be less sensitive to missing observations and the number of bonds and bills used in the estimation.

In practice our criteria of evaluation boils down to comparing measures of fit and convergence properties for NS and LS.

The LS functional form is derived in a theoretical model. We would like to emphasize that we do not attempt to test the theoretical model, for instance whether the restrictions it imposes are empirically fulfilled or not. We simply use

² Svensson (1993a) compares estimation of forward rates with simpler approximate methods to estimation with the Longstaff and Schwartz functional form. The simpler methods are the "uncorrected" method of approximating a coupon bond by a zero-coupon bond with the same maturity, the "duration" method of approximating a coupon bond by zero-coupon bond with the same duration, and the "recursive method with interpolation." The last method involves interpolating yields to maturity and coupons to find synthetic coupon bonds that mature exactly one year apart, in which case a convenient recursive formula can be used to compute spot and forward rates. The uncorrected and duration methods imply a bias in estimating forward interest rates, and the recursive method with interpolation relies on a non-convincing assumption of continuity of coupons in time to maturity. The simpler methods appear clearly inferior to estimation of the discount function with either a cubic spline or the NS or LS functional forms.

the functional form to fit it to the data, without testing whether the theoretical restrictions are fulfilled.³

The practical difference between the NS and LS functional forms is that the NS is much easier to use whereas LS is much more flexible. The result of our comparison is briefly as follows. The NS and LS estimates of spot and forward rates are very similar. LS has a marginally better fit than NS. The NS fit is well within the precision that seem reasonable for monetary policy analysis, though. With regard to convergence and computation, NS is child's play, whereas LS is close to a nightmare. For the sample studied NS therefore from a practical point of view appears much superior to LS. Put differently, the Swedish term structure studied is not so complicated that the flexibility of LS is needed. Estimates in Svensson (1993c) indicate that NS performs satisfactory also for the term structure in Britain, France, Germany, and the United States from September 1992 to September 1993. The flexibility of the LS functional form would be needed only for a very complex term structure when the fit of NS is bad.

The paper is outlined as follows. Section 2 provides definitions and describes the method to estimate spot and forward spot and forward rates in general terms. Section 3 and 4 present the NS and LS functional forms. Section 5 presents the data and discusses the method of comparison and the details of the estimation. Section 6 reports the results and section 7 concludes.

³ For instance, we deviate from the theory in estimating parameters separately for each trade date, whereas according to the theory the parameters should be constant across dates.

2 Estimation of Spot and Forward Rates

First we restate definitions and the simple algebra of yields to maturity, spot rates and forward rates (see for instance Shiller (1990)). Consider a coupon bond with a principal of 100 Swedish kronor, an annual coupon c (measured as a proportion of the principal); a time to maturity m (measured in years), and a price p in Swedish kronor (net of accrued interest rate). The annually compounded yield to maturity is the annually compounded internal rate of return that makes the present value of the coupon payments and principal equal to the price of the bond.

Formally, let $\tau_k, k = 1, 2, \dots, K$, denote the times for the coupon payment, where K is the number of coupon payments. In the special case when m is an integer, we simply have $\tau_k = k$ and $K = m$. In the general case we have

$$\tau_k = m - [m] + k - 1 \text{ and } K = [m] + 1, \quad (2.1)$$

where $[m]$ denotes the largest integer that is strictly smaller than m . The yield to the maturity and the price of the bond are then related according to

$$p \equiv \sum_{k=1}^K \frac{100c}{(1+y)^{\tau_k}} + \frac{100}{(1+y)^{\tau_K}} \quad (2.2)$$

(The last term on the right hand side is the present value of the principal of the bond.)

Let $d(m)$ denote the price of a zero-coupon bond with principal 1 krona and time to maturity m years. The spot rate is the yield to maturity on a zero-coupon bond. The algebra of spot and forward rate is easiest if spot and forward rates are expressed as continuously compounded rates.⁴ Then the spot rate $s(m)$ and the price of the discount bond $d(m)$ are related according to

$$d(m) \equiv \exp[-s(m)m] \text{ and } s(m) \equiv -\frac{\ln d(m)}{m} \quad (2.3)$$

Let $f(m, M)$ denote the (implied) forward rate with settlement in m years and maturity in $M > m$ years. It fulfills

⁴ Continuously compounded interest rates s and annually compounded rates \bar{s} are related according to $s = \ln(1 + \bar{s})$ and $\bar{s} = \exp(s) - 1$.

$$f(m, M) \equiv -\frac{\ln d(M) - \ln d(m)}{M - m} \equiv \frac{s(M)M - s(m)m}{M - m}. \quad (2.4)$$

The instantaneous forward rate $f(m)$ with settlement (and maturity) in m years is defined as

$$f(m) \equiv \lim_{M \rightarrow m} f(m, M) \equiv -\frac{\partial \ln d(m)}{\partial m} \equiv s(m) + m \frac{\partial s(m)}{\partial m}. \quad (2.5)$$

It follows that the spot rate for a given maturity is the average of the instantaneous forward rates with settlement between zero and the spot rate's maturity.

$$s(m) \equiv \frac{1}{m} \int_{\tau=0}^m f(\tau) dt, \quad (2.6)$$

Finally, let $s(0)$ and $f(0)$ denote the limits of the spot rate $s(m)$ and instantaneous forward rate $f(m)$ when the maturity m approaches zero ($s(0)$ is the instantaneous spot rate). It follows from (2.6) that they are equal,

$$s(0) = f(0) \quad (2.7)$$

The relations (2.5)-(2.6) imply that spot rates and instantaneous forward rates are related exactly as average and marginal cost of production, where the time to maturity corresponds to quantity produced.

The problem to estimate spot and forward rates can then be stated as follows. For a given trade date, let there be n coupon bonds, where bond $j = 1, \dots, n$, is represented by the triple (c_j, m_j, p_j) of the coupon c_j , the time to maturity m_j , and the observed price p_j . (If observed yields to maturity rather than prices are available, the observed prices are calculated from (2.2).) Let the discount function be modeled by a particular functional form $d(m; b)$, where b is a vector of parameters. The model price of each bond (net of accrued interest), $P_j(b)$, is the present value of the bond when the coupon payments and the principal value are priced with the discount function,

$$P_j(b) \equiv \sum_{k=1}^{K_j} 100 \cdot c_j d(\tau_{jk}; b) + 100 \cdot d(\tau_{jK_j}; b), \quad j = 1, \dots, n, \quad (2.8)$$

where $\tau_{jk}, k = 1, \dots, K_j$ denotes the times of the coupon payments on bond j .

The observed price is assumed to differ from the model price by an error term with zero expectations

$$p_j = P_j(b) + \varepsilon_j, E[\varepsilon_j] = 0. \quad (2.9)$$

The error term can be motivated by institutional features. The yield spread in the data base from Sveriges Riksbank that we use is constructed by taking the best bid and the best ask yield at closing time, hence constructing a minimum bid-ask spread. This procedure may incorporate some mispricing. For example, the yields collected do not necessarily reflect trades at the same time. In addition, yield volatility is usually especially high at closing time, perhaps due to temporary imbalances in supply and demand.^{5 6}

The model prices are then fitted to the actual prices with non-linear least squares or with maximum-likelihood (assuming error terms are normal). That is, the estimate \hat{b} is given by

$$\hat{b} = \arg \min_b \sum_{j=1}^n [P_j(b) - p_j]^2 \quad (2.10)$$

This approach has been used by several authors, with different functional forms for the discount function.⁷ McCulloch (1971, 1975) used a quadratic and cubic spline, respectively. The latter has become a standard method. McCulloch's formulation has the advantage that the estimation can be expressed as a simple linear regression. Cubic spline estimates often leads to very unstable forward rates, though; especially for the longest maturities in the sample for which the forward rates may be either very large or very small, sometimes even negative (Shea (1984), Langetieg and Smoot (1989), see also the graphs in McCulloch (1990) or the Gauss viewing program that comes with McCulloch and Kwon (1993)). This is a drawback, especially if the focus is on the forward rate estimates, as in monetary policy analysis. The same problem arises for exponential splines (Vasicek and Fong (1982), Shea (1985)). Carleton and Cooper (1976) estimated zero-coupon prices without any restriction on continuity, which implied large fluctuations in spot and forward rates. Chambers, Carleton and Waldman (1984) used a polynomial for the spot and

⁵ Our maintained assumption is that the given functional form is the true form for the discount function, and that only the parameters are unknown and remain to be estimated. Green and Oedegaard (1993) interpret the error term as partly a specification error because the true functional form may deviate from the assumed functional form.

⁶ Below we shall see that the mean absolute price error is not larger than the average bid-ask spread in the market.

⁷ Bank of England estimates a par yield curve with a different method, namely by fitting a surface to bond data in the yield, coupon and time-to-maturity plane (see Bank of England (1990) and Mastronikola (1991)).

forward rates, which however implies that spot and forward rates for long maturities reach large positive or negative values.⁸

From an economics point of view it seems reasonable, though, that spot and forward rates for long maturities should be positive and approximately constant. The property that spot and forward rates approach a constant for long maturities is shared by several recently suggested functional forms. Nelson and Siegel's (1987) simple functional form has this property, as has several functional forms that are derived from equilibrium models, for instance the one-state-variable model of Cox, Ingersoll and Ross (1985), or the two-state-variable model of Longstaff and Schwartz (1992).^{9 10}

As stated above, in this paper we shall compare estimation with the simple form of Nelson and Siegel with the complex form of Longstaff and Schwartz. Next we therefore give the details of these functional forms.^{11 12}

⁸ Schaefer (1981) uses Bernstein polynomials which avoids the problem of negative forward rates.

⁹ The models of Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992) can be seen as special cases of the n -state-variable model of Duffie and Kan (1993).

¹⁰ Tanggaard (1992) suggest a nonparametric kernel smoothing procedure to estimate the discount function.

¹¹ Majnoni (1993) compares the functional forms of Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992) on Italian data.

¹² As mentioned we do not test the theoretical model of Longstaff and Schwartz (1992). The one-state-variable model of Cox, Ingersoll and Ross (1985) has been subject to several tests, for instance by Brown and Dybvig (1986), Gibbons and Ramaswamy (1986) and Brown and Schaefer (1993).

3 Nelson & Siegel

Nelson and Siegel (1987) assume that the instantaneous forward rate is the solution to a second-order differential equation with two equal roots. Hence it can be written

$$f(m; b) \equiv \beta_0 + \beta_1 \exp\left(-\frac{m}{\tau}\right) + \beta_2 \frac{m}{\tau} \exp\left(-\frac{m}{\tau}\right), \quad (3.1)$$

where $b = (\beta_0, \beta_1, \beta_2, \tau)$ is the vector of parameters. The spot rate can be derived by integrating the forward rate. It is given by

$$s(m; b) \equiv \beta_0 + (\beta_1 + \beta_2) \frac{1 - \exp\left(-\frac{m}{\tau}\right)}{\frac{m}{\tau}} - \beta_2 \exp\left(-\frac{m}{\tau}\right). \quad (3.2)$$

The spot and forward rates have convenient properties. The limit of the spot and forward rates when maturity approaches infinity and zero, respectively, are

$$f(\infty; b) = s(\infty; b) = \beta_0 \text{ and} \quad (3.3)$$

$$f(0; b) = s(0; b) = \beta_0 + \beta_1. \quad (3.4)$$

Thus the spot and forward rate approach a constant for long maturities and settlements.

Furthermore, suppose there exist a stationary point for the forward rate. That is, suppose there exists $\hat{m} \geq 0$ such that $\frac{\partial f(\hat{m}; b)}{\partial m} = 0$, and let $\hat{f} = f(\hat{m}; b)$. Then

$$\hat{f} = \beta_0 + \beta_2 \exp\left(-1 + \frac{\beta_1}{\beta_2}\right) \text{ and} \quad (3.5)$$

$$\frac{\hat{m}}{\tau} = 1 - \frac{\beta_1}{\beta_2}. \quad (3.6)$$

Thus, given β_0 and β_1, β_2 is determined by (3.5) if there exists a maximum or a minimum \hat{f} . Furthermore, since the second derivative of $f(m; b)$ at \hat{m} fulfills

$$\frac{\partial^2 f(\hat{m}, b)}{\partial m^2} = -\beta_2 \frac{1}{\tau^2} \exp\left(-\frac{\hat{m}}{\tau}\right). \quad (3.7)$$

It follows that the sign of β_2 determines whether there is a maximum or a minimum, a negative (positive) β_2 corresponds to a minimum (maximum). Finally, given β_1 and β_2 , τ is determined by \hat{m} according to (3.6).

It follows that $\beta_0, \beta_1, \beta_2$, and τ are determined recursively in order by $f(\infty), f(0), \hat{f}$, and \hat{m} . The parameters are therefore rather intuitive, and it is easy to find suitable starting values for the optimization procedure.

The Nelson and Siegel discount function is then given by

$$d(m; b) \equiv \exp[-s(m; b)m], \quad (3.8)$$

where $s(m; b)$ is given by (3.2).

The Nelson and Siegel forward rate is a very simple functional form. It can have at most one stationary point.

4 Longstaff & Schwartz

Longstaff and Schwartz (1992) specify a model where there are two state variables, the instantaneous spot rate r and the spot rates instantaneous rate of variance V . The two state variable are assumed to follow mean-reverting stochastic processes. Their model can be seen as a version with two state variables of Cox, Ingersoll and Ross's (1985) model with one state variable. Longstaff and Schwartz then derive an equilibrium discount function as a solution to a partial differential equation. The solution is a very complex functional form,

$$F(m, r, V) \equiv A(m)^{2r} B(m)^{2\eta} \exp[\kappa m + C(m)r + D(m)V], \quad (4.1)$$

where

$$A(m) \equiv \frac{2\varphi}{(\delta + \varphi)(\exp(\varphi m) - 1) + 2\varphi}, \quad (4.2)$$

$$B(m) \equiv \frac{2\psi}{(\nu + \psi)(\exp(\psi m) - 1) + 2\psi}, \quad (4.3)$$

$$C(m) \equiv \frac{\alpha\varphi(\exp(\psi m) - 1)B(m) - \beta\psi(\exp(\varphi m) - 1)A(m)}{\varphi\psi(\beta - \alpha)}, \quad (4.4)$$

$$D(m) \equiv \frac{\psi(\exp(\varphi m) - 1)A(m) - \varphi(\exp(\psi m) - 1)B(m)}{\varphi\psi(\beta - \alpha)}, \quad (4.5)$$

$$\varphi = \sqrt{2\alpha + \delta^2}, \quad (4.6)$$

$$\psi = \sqrt{2\beta + \nu^2} \text{ and} \quad (4.7)$$

$$\kappa = \gamma(\delta + \varphi) + \eta(\nu + \psi). \quad (4.8)$$

The parameters $\alpha, \beta, \gamma, \eta, \delta$ and ν are functions of the parameters of the stochastic processes for the state variables r and V and the investors' risk aversion.¹³ The parameters must be nonnegative, except ν which may be of either sign. The state variables and the parameters α and β must fulfill the restrictions.

¹³ The parameter ν is the sum of a parameter, λ , which is proportional to the market price of risk and may be of either sign, and a nonnegative parameter, ζ , in the underlying diffusion process.

$$\alpha < \frac{V}{r} < \beta \text{ or } \beta < \frac{V}{r} < \alpha. \quad (4.9)$$

Considered as a functional form for the discount function as a function of the time to maturity, the state variables are also regarded as parameters and the parameter vector is hence $b = (r, V, \alpha, \beta, \gamma, \eta, \delta, \nu)$. The discount function is hence given by

$$d(m; b) \equiv F(m; r, V, \alpha, \beta, \gamma, \eta, \delta, \nu). \quad (4.10)$$

The spot rate can be found from (2.3) and is given by

$$s(m; b) = -\frac{\kappa m + 2\gamma \ln A(m) + 2\eta \ln B(m) + C(m)r + D(m)V}{m}. \quad (4.11)$$

The instantaneous forward rate can be derived according to (2.5). The spot and forward rates have the properties

$$f(0; b) = s(0; b) = r \text{ and} \quad (4.12)$$

$$f(\infty; b) = s(\infty; b) = \gamma(\varphi - \delta) + \eta(\psi - \nu). \quad (4.13)$$

Thus the spot and forward rates approach a constant for long maturities and settlements.

The Longstaff and Schwartz functional form has some theoretical support, since it is derived from an equilibrium model. The form is very flexible and the forward rate can have both a maximum and a minimum.

5 Data and Method

The data consists of daily data from November 23, 1992, (two business days after the krona was floated on November 19, 1992) to June 21, 1993, 142 trade dates altogether. Each trade date has observations of the so-called marginal lending rate (the rate at which Sveriges Riksbank lends overnight reserves to banks), the yields on the 11-12 outstanding Swedish Treasury Bills, and the yields on the 6-7 outstanding Government Benchmark Bonds (the longest maturity was 10 years until January 1993, when a new 16-year bond was issued). Arbitrage on the interbank overnight rate makes the interbank overnight rate close to the marginal lending rate.

The parameters of the discount function were estimated for each trade date separately. That is, the parameters were allowed to change between trade dates. Three different cases, denoted NS1, NS2 and LS were estimated.

Case NS1 is the estimation of the Nelson and Siegel discount function when the spot and forward rates for zero maturity/settlement, $s(0)$ and $f(0)$, are restricted to be equal to the marginal lending rate. This restriction was imposed to make the estimates comparable to the estimates with case LS, where this restriction was also imposed (see below). In practice NS1 amounts to impose the restriction that the sum of the parameters β_0 and β_1 equal the marginal lending rate.

Case NS2 is the estimation of the Nelson and Siegel discount function without the above restriction, in which case $s(0)$ and $f(0)$ may deviate from the marginal lending rate.

Case LS is the estimation of Longstaff and Schwartz discount function. The discount function then has 8 parameters, the two state variables r and V and the 6 parameters $\alpha, \beta, \gamma, \eta, \delta$ and ν . Since the two state variables can be interpreted as the overnight rate and the volatility of the overnight rate, respectively, they can in principle be estimated separately, or estimated jointly with the other parameters. Because of convergence difficulties and indications that the functional form is overparameterized, it is preferable to estimate the state variable separately, in order to reduce the number of parameters to be estimated jointly. We simply imposed the restriction that r equals the overnight rate. As for the volatility V we tried to estimate it as a GARCH process for the volatility of 1-week, 1-month and 3-month Treasury bill rates.¹⁴ We could not reject the hypothesis that the volatility was constant, however. Therefore, throughout the sample we restricted V to equal 0.0010 per year³, 10 basis

¹⁴ Since the marginal lending rate is held constant between the instances at which it is changed by the central bank, we thought that the volatility in the LS model better corresponds to the volatility of short rates with maturities between 1 week and 3 months.

points per year³, which is about the average volatility for these rates.¹⁵ Thus we ended up estimating the remaining six parameters $\alpha, \beta, \gamma, \eta, \delta$ and ν , taking into account the nonnegativity constraints on all except ν , and taking into account the restriction (4.9).

The restriction that the estimated spot and forward rate should go through the marginal lending or the overnight rate can be motivated by the fact that fitting model prices to observed bond and bill prices gives a low weight to the fit of short term yields, since the prices are insensitive to the yields for short maturities. The restriction can be seen as a way of compensating for the low weight in the fit given to the short yields. In our case the low weight is to some extent compensated for because we have relatively many T-bills and relatively few bonds in the Swedish sample. Another possibility would be to experiment by imposing weights on the errors between model and observed prices that decrease at different rates with the time to maturity. Parameters could then be estimated by weighted Nonlinear Least Squares.¹⁶

NS1 and NS2 have been estimated with Maximum Likelihood, including heteroskedasticity-consistent ML estimates of the covariance matrix of the parameters. For LS we usually encountered difficulties in the computation of the covariance matrix for the parameters, probably due to flatness of the objective function near the optimum. The reported LS estimates have then been estimated with Nonlinear Least Squares, which here gives the same point estimates as ML.

As mentioned, the evaluation of the three different cases is then done in terms of measures of fit and convergence properties.

¹⁵ Since interest rates have the dimension per year, the variance of an interest rate has the dimension per year², and V , the instantaneous rate of variance of an interest rate, has the dimension per year³. The volatility of the 1-week, 1-month and 3 month were between 8 and 12 basis points per year².

¹⁶ See for instance Coleman, Fisher and Ibbotson (1992), Langetieg and Smoot (1989) and Majnoni (1993) for examples of different weighting of the error terms.

6 Results

We start by discussing two examples, the estimates for the trade dates November 23, 1992, and April 16, 1993. *Figures 1a-c* shows actual yields to maturity and estimated spot and forward rates for November 23, 1992, estimated for NS1 (Nelson and Siegel with spot and forward rates for zero maturity restricted to equal the marginal lending rate), NS2 (Nelson and Siegel without the above restriction) and LS (Longstaff and Schwartz with the above restriction). The squares are observed yields to maturity (percent per year, annually compounded) for the marginal lending rate, 12 Treasury bills and 6 Government bonds. The dashed curves show the estimated spot rates, the solid curves the estimated instantaneous forward rates, and the thin horizontal dashed lines show the infinite-maturity spot and long rate (the horizontal asymptote $s(\infty)$ and $f(\infty)$). The error bars in the NS diagrams are 95 percent confidence intervals, computed with the delta method.¹⁷

Figures 2a-c show, for NS1, NS2 and LS, observed T-bill and bond prices (squares), estimated prices (dots) (error bars for NS1 and NS2 denote with 95 percent confidence intervals), and coupons (pluses), for the same trade date, November 23, 1992.

Figures 3a-c and *4a-c* show the same things for the trade date April 16, 1993.

Table 1 reports the parameter estimates and measures of the fit for the two dates. Standard errors are included for NS1 and NS2. We see that the NS parameters are fairly precisely estimated. The resulting confidence intervals for the spot and forward rates are also rather narrow, as seen in the figures.

It is apparent from the figures that the NS and LS estimates are very similar. Both spot and forward rates are close. For spot rates the largest differences occur at the shortest end, and depend upon whether the NS estimation is restricted to coincide with the marginal borrowing rate or not. For forward rates the differences are evenly distributed across different settlements even though the differences on November 23 are more distinct for longer maturities. For April 16 the LS estimates are within the confidence interval of the NS estimates. In *Figures 2a-c* the fit of the estimated prices is very good in all cases and differences between the estimates are hardly visible. The Root Mean Squared Price Errors (RMSPEs) in the three cases vary between 5 and 8 basis points (hundredths of percent of the principal of the bond); the Mean Absolute Price Errors (MAPEs) vary between 4 and 6 basis points. For the yields, the

¹⁷ Although the algebra is easiest with continuously compounded rates, the figures report annually compounded rates, since yields on bonds with annual coupons usually are reported as annually compounded rates.

RMSYEs vary between 9 and 20 basis points per year; the MAYEs vary between 6 and 12 basis points per year.

The fit of LS is marginally better. In terms of RMSPE the fit of NS2 (without the restriction) is naturally better than that of NS1 (with the restriction). The difference is rather small though. The yield errors are similar, except for NS2 on November 23. This is due to the errors for the shortest maturities, as is apparent from Figure 1b. Whether or not the restriction is imposed in the NS estimation indeed sometimes leads to large differences (sometimes 100 basis points per year) in estimated spot and forward rates for the shortest maturities. This is not difficult to understand, since minimizing price errors give little weight to short yields, since prices are insensitive in short yields.

The differences between the estimates on November 23 are larger than typical, whereas the differences between the estimates on April 16 are typical for the rest of the sample.

Finally, we see in Table 1 that the convergence properties are rather different. NS1 and NS2 converge with few iterations in short time. LS needs many more iterations, and even more time, since each iteration takes longer.

Next we discuss summary results for the whole sample. *Table 2* summarizes the results for the fit for NS1, NS2 and LS, respectively. *Table 3* reports summary statistics for absolute differences in estimated spot and forward interest rates between NS1 and LS. *Table 4* summarizes the results for the parameter estimates and the convergence properties.

Figures 5a-c show estimated spot rates for the whole sample period, for NS1, NS2, and LS, and *Figures 6a-c* show corresponding forward rates.

With regard to the fit, in Table 2 we see that the (sample) mean of the MAPE and RMSPE are highest for NS1, 8 and 10 basis points respectively, with (sample) standard deviations 3 basis points and (sample) maxima 17 and 19 basis points. The mean of the MAPE and RMSPE are lower and similar for NS2 and LS, about 5 and 7.5 basis points, respectively, with standard deviations between 1 and 3 basis points, and maxima between 9 and 16 basis points. The mean of the MAYEs vary between 7 and 10 basis points per year, with standard deviation between 3 and 5 basis points per year, lowest for LS. Altogether LS appears to have a marginally better fit than NS1 but no better fit than NS2. The maximum MAPEs does not exceed 20 basis points, and the maximum MAYE does not exceed 40 basis points per year, which indicate a precision more than sufficient for monetary policy analysis.

With regard to the absolute deviations in estimated spot and forward rates, reported in Table 3, we see that the estimated spot rates are very similar. The

means for the differences in the spot rates with 1-10 years to maturity are between 2 and 5 basis points per year. The largest differences occur at the shorter maturities. The maxima are about 20 basis points per year for maturities longer than one year. For the forward rates, the average differences are slightly larger, 7 to 11 basis points per year, and they seem to be uniformly distributed over the maturity. The standard deviations are larger and the greater variability can also be seen in the minimum and maximum columns. However, the maxima do not exceed 40 basis points per year for maturities less than 10 years.

With regard to the parameter estimates, we see in Table 4 that for NS1 and NS2 the means of β_0 , β_1 and β_2 are about 10, -0.4 and -6 percent per year (the dimension of the betas). (The mean of $f(\infty)$ exceeds the mean of β_0 for NS1 and NS2 simply because the first is annually and the second is continuously compounded). The corresponding standard deviations are between 0.4 and 1.3 percent per year. The mean of τ is about 1.4 years (the dimension of τ with standard deviation of about 0.3 year).

For the parameters of LS, the standard deviations are much larger relative to the means. The estimates of the parameters are in that sense rather unstable. This also shows in that similar estimated spot and forward curves have rather different parameters. This indicates that the correlation between the estimates may be high, and that the model may be overparameterized. This has not been possible to verify by estimating the covariance matrix of the estimates for LS. However, the circumstance that the covariance matrix is difficult to compute is in itself an indication of overdetermination. However, the sample covariance matrix of the LS estimates (not reported) does *not* indicate high correlation between the parameters.

There is of course no presumption that the parameters should be constant during the sample. Therefore, the means reported in Table 4 are not estimates of constant parameters, they only indicate the average magnitude of the parameters. Nevertheless, if the sample means of the NS and LS parameters would be interpreted as estimates of constant parameters over the sample, and if each observation would be assumed to be independent, corresponding t -statistics would indicate that all parameters are significantly different from zero. (The standard errors are then the square root of the sample variance divided by the number of observations less one.)

The (sample) mean of the asymptote $f(\infty)$ is about 100 basis points per year lower for LS than for NS1 and NS2. The (sample) standard deviation is larger for LS than for NS1 and NS2: 43 basis points per year for NS1 and NS2, and almost 8 times larger for LS. This result arises since in a few cases the LS estimation results in a very low estimate of $f(\infty)$. Consistent with this, the medians of the estimates of $f(\infty)$ are similar, 10.65, 10.66 and 10.60 percent per year for NS1, NS2 and LS, respectively.

The parameter estimates of NS, including the asymptotic spot and forward rate $f(\infty)$, are much more stable than those of LS. This, together with their clear interpretation, is certainly an advantage for NS.

With regard to the convergence properties, the convergence for both NS1 and NS2 is relatively insensitive to starting values and occurs with relatively few iterations. Each iteration is also quick, so in general convergence is fast and easy. NS1 converges even more easily than NS2, on average 16 against 28 iterations, and maximum 38 iterations against 250. The average time for each iteration is about 0.5 second on a 486 machine with 50 MHz clock frequency. Using NS is from this point of view child's play.

In contrast, convergence in LS is extremely sensitive to starting values, and frequently requires very many iterations. Many iterations are very slow. Local minima abound. The average number of iterations is more than 500, sometimes 2000 iterations are required. The average time for each iteration is about 1.5 seconds, but frequently each iteration takes up to 8 seconds. In 8 cases convergence failed. Compared to NS, convergence in LS is close to a nightmare.

Obviously, in terms of convergence properties, the advantage of NS is huge.

7 Conclusions

We have estimated the Swedish term structure with two functional forms, the simple form of Nelson and Siegel (1987) and the complex form of Longstaff and Schwartz (1992). The functional forms have been compared with regard to their performance in estimating spot and forward interest rates to be used in monetary policy analysis, for instance as monetary policy indicators.

The result of our comparison is that LS has a marginally better fit, but that NS is superior with regard to convergence properties, confidence interval computation, parameter stability, and parameter interpretation. Furthermore, the NS fit seems well above what is needed for monetary policy analysis. On balance, our comparison thus favors NS.

The comparison is made on the Swedish term structure between November 1992 and June 1993. It appears that this term structure was not sufficiently complicated to warrant the flexibility of LS. This does of course not exclude the possibility that the term structure on other occasions and for other countries could be too complicated for NS and therefore warrant LS or other more flexible forms. Estimation of the term structure for Britain, France, Germany and the United States for six selected dates between September 1992 and September 1993 in Svensson (1993c) indicates, however, that NS gives a satisfactory fit also for these countries and this period.¹⁸ Since the NS functional form only allows for one interior maximum or minimum, one major determinant of whether the NS gives a good fit or not should be whether the term structure has more than one interior maximum or minimum. It remains an open question how often that occurs.¹⁹

In any case, a simple operational way to estimate the term structure is to start with a simple form like NS and then judge whether the fit is sufficiently good. If not, a more complex form should be tried to see if the fit improves.

¹⁸ The fit for Germany is sometimes not as good as for the other countries.

¹⁹ It is of course possible to compare estimation with NS and LS in a Monte-Carlo study. However, the result of the comparison will of course be heavily influenced by what functional form is used to generate the data.

Table 1. Estimation Results, November 23, 1992, and April 16, 1993

Parameters	NS1		NS2		LS	
	(1) Nov 23	(2) Apr 16	(3) Nov 23	(4) Apr 16	(5) Nov 23	(6) Apr 16
β_0 (%/yr)	10.24 (0.05)	10.29 (0.03)	10.32 (0.05)	10.30 (0.04)		
β_1 (%/yr)	2.26	-0.54	1.58 (0.24)	-0.59 (0.13)		
β_2 (%/yr)	-5.76 (0.24)	-6.16 (0.06)	-5.11 (0.31)	-6.08 (0.19)		
τ (yrs)	0.87 (0.04)	1.53 (0.04)	1.04 (0.10)	1.55 (0.07)		
α					0.0061	0.0000
β					0.0102	0.0260
γ					8.4211	0.0042
δ					2.5153	1.0899
η					2.6390	1.4348
ν					0.3001	0.3234
r (%/yr)					12.50	9.75
V (b.p./yr ³)					10.00	10.00
Measures of fit						
RMSPE (%)	0.0822	0.0673	0.0656	0.0669	0.0532	0.0620
MAPE (%)	0.0580	0.0516	0.0485	0.0515	0.0432	0.0474
RMSYE (%/yr)	0.1160	0.0889	0.2031	0.1003	0.0940	0.0971
MAYE (%/yr)	0.0756	0.0622	0.1155	0.0683	0.0630	0.0615
Convergence						
Iterations	10	18	14	15	501	377
Time (minutes)	0.05	0.14	0.10	0.14	10.43	10.32

Note: NS1, NS2 and LS refer to Nelson and Siegel with restriction, without restriction and Longstaff and Schwartz with restriction, respectively. Heteroskedasticity-consistent standard errors for the parameters in column (1)-(4) are given in parentheses. RMSPE and MAPE denote the mean root square price error and mean absolute price error, respectively, in percent of the principal. RMSYE and MAYE are the analogs for yield errors, in percentage points per year. Iterations and Time denote the number of iterations and the time to convergence.

Table 2. Summary of Fit**2a. NS1 (Nelson & Siegel with restriction)**

	Mean	Std Dev	Min	Max	Obs	Miss
RMSPE	0.1010	0.0278	0.0406	0.1912	142	1
MAPE	0.0760	0.0256	0.0315	0.1751	142	1
RMSYE	0.1409	0.0722	0.0335	0.3646	142	1
MAYE	0.0972	0.0481	0.0253	0.2628	142	1

2b. NS2 (Nelson & Siegel without restriction)

	Mean	Std Dev	Min	Max	Obs	Miss
RMSPE	0.0751	0.0255	0.0202	0.1594	142	1
MAPE	0.0483	0.0121	0.0140	0.0870	142	1
RMSYE	0.1537	0.0986	0.0295	0.3928	142	1
MAYE	0.0813	0.0436	0.0240	0.2060	142	1

2c. LS (Longstaff & Schwartz with restriction)

	Mean	Std Dev	Min	Max	Obs	Miss
RMSPE	0.0738	0.0312	0.0200	0.1584	142	8
MAPE	0.0508	0.0195	0.0155	0.1265	142	8
RMSYE	0.1143	0.0599	0.0262	0.3105	142	8
MAYE	0.0682	0.0327	0.0227	0.2103	142	8

Note: RMSPE and MAPE (in percent of the principal) denote the root mean square price error and mean absolute price error, respectively. RMSYE and MAYE (in percentage points per year) are the analogs for yield errors. Obs refers to the total number of trade dates, and Miss refers to the number of trade dates for which convergence failed.

Table 3. Summary of Absolute Differences in Spot and Forward Rates

3a. Spot Rates (NS1-LS)

Maturity	Mean	Std Dev	Min	Max	Obs	Miss
0.5	0.0973	0.0835	0.0008	0.3793	134	8
1	0.0485	0.0460	0.0003	0.2028	134	8
3	0.0414	0.0326	0.0004	0.1468	134	8
5	0.0192	0.0169	0.0003	0.0681	134	8
7	0.0221	0.0192	0.0000	0.0730	134	8
10	0.0187	0.0145	0.0002	0.0569	134	8

3b. Forward Rates (NS1-LS)

Settlement	Mean	Std Dev	Min	Max	Obs	Miss
0.5	0.0763	0.0673	0.0024	0.3080	134	8
1	0.0812	0.0969	0.0001	0.3901	134	8
3	0.0702	0.0642	0.0009	0.3401	134	8
5	0.1014	0.0787	0.0002	0.3133	134	8
7	0.0649	0.0521	0.0002	0.2527	134	8
10	0.1135	0.1501	0.0008	0.6452	134	8

Note: Summary statistics for absolute differences (in percentage points per year) in estimated spot and forward rates. NS1 and LS refer to Nelson and Siegel with restriction and Longstaff and Schwartz with restriction, respectively. Maturity and settlement is measured in years. Obs refers to the number of trade dates for which convergence occurred for both NS1 and LS, Miss refers to the number of trade dates for which convergence failed for either NS1 or LS.

4. Summary of Parameter Estimates and Convergence

4a. NS1 (Nelson & Siegel with restriction)

	Mean	Std Dev	Min	Max	Obs	Miss
β_0 (%/yr)	10.11	0.39	9.43	10.79	142	1
β_2 (%/yr)	-5.92	0.91	-7.23	-3.02	142	1
τ (yrs)	1.31	0.30	0.56	1.94	142	1
$f(\infty)$ (%/yr)	10.64	0.43	9.89	11.40	142	1
Iterations	16.05	4.51	8.00	34.00	142	1

4b. NS2 (Nelson & Siegel without restriction)

	Mean	Std Dev	Min	Max	Obs	Miss
β_0 (%/yr)	10.16	0.39	9.46	11.22	142	1
β_1 (%/yr)	-0.42	1.00	-1.61	3.59	142	1
β_2 (%/yr)	-5.47	1.26	-8.13	-3.19	142	1
τ (yrs)	1.46	0.26	0.48	2.01	142	1
$f(\infty)$ (%/yr)	10.70	0.43	9.92	11.87	142	1
Iterations	27.70	22.65	13.00	205.00	142	1

4c. LS (Longstaff & Schwartz with restriction)

	Mean	Std Dev	Min	Max	Obs	Miss
α	0.01	0.02	0.00	0.17	134	8
β	0.05	0.05	0.00	0.22	134	8
γ	2.81	5.64	0.00	27.10	134	8
δ	1.43	1.21	0.00	5.56	134	8
η	1.67	3.07	0.00	13.76	134	8
ν	0.53	1.04	-0.94	3.83	134	8
$f(\infty)$ (%/yr)	9.60	3.20	0.00	14.91	134	8
Iterations	538.34	436.60	32.00	2458.00	134	8

Note: Obs refers to the total number of trade dates, and Miss refers to the number of trade dates for which convergence failed.

Figure 1a. NS1 (Nelson and Siegel, w/ restriction), Nov 23, 1992
95% confidence interval

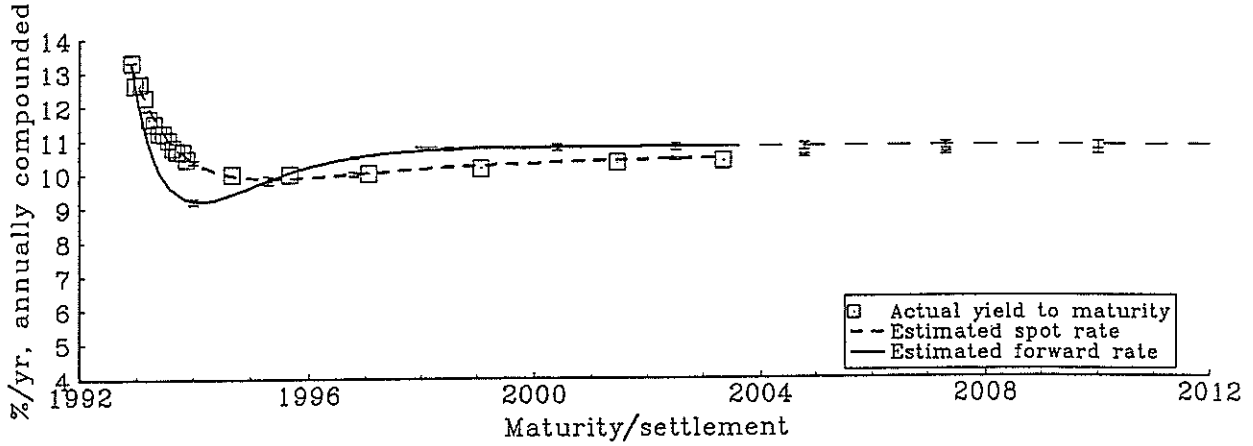


Figure 1b. NS2 (Nelson and Siegel, w/o restriction), Nov 23, 1992
95% confidence interval

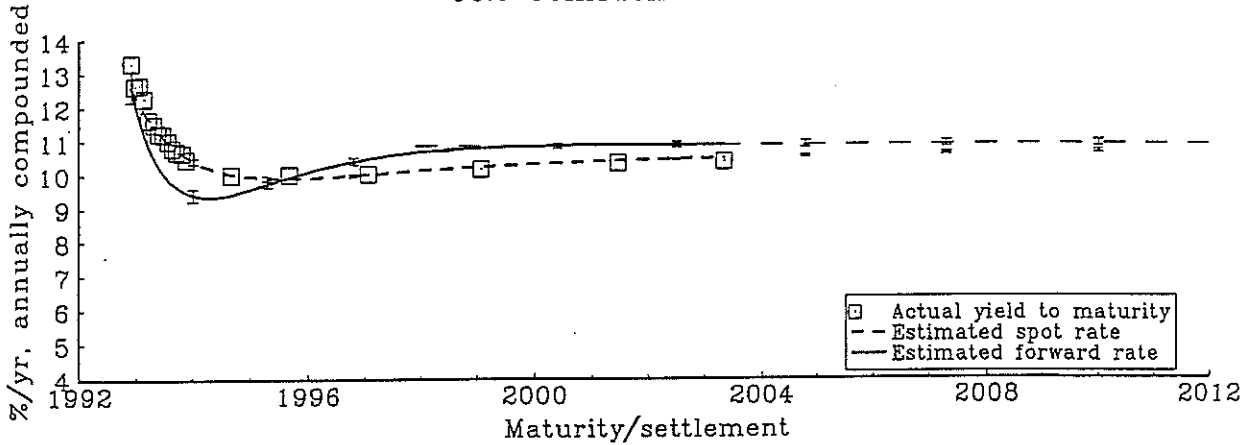


Figure 1c. LS (Longstaff and Schwartz, w/ restriction), Nov 23, 1992

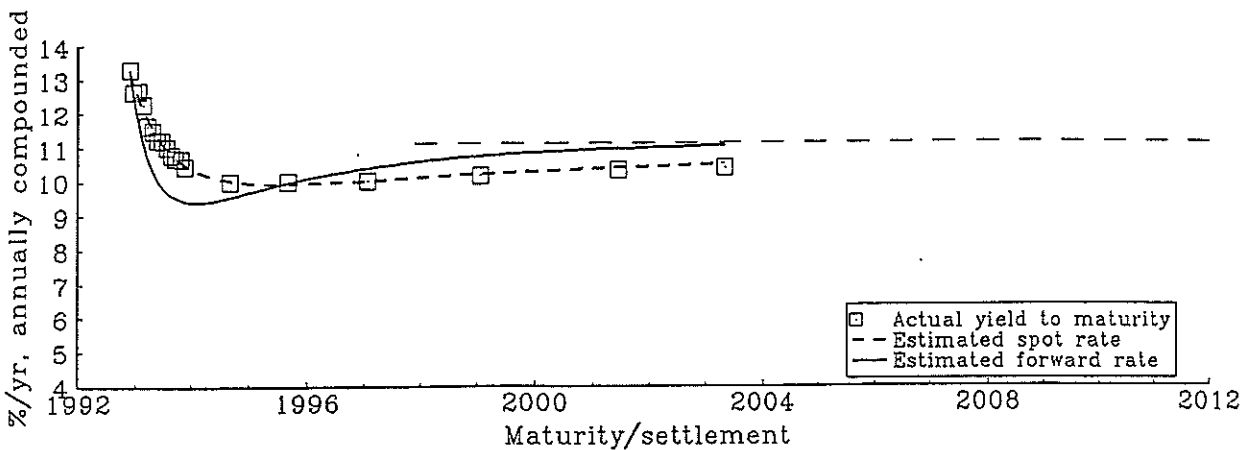


Figure 2a. NS1, Actual and Estimated Bond Prices, Nov 23, 1992

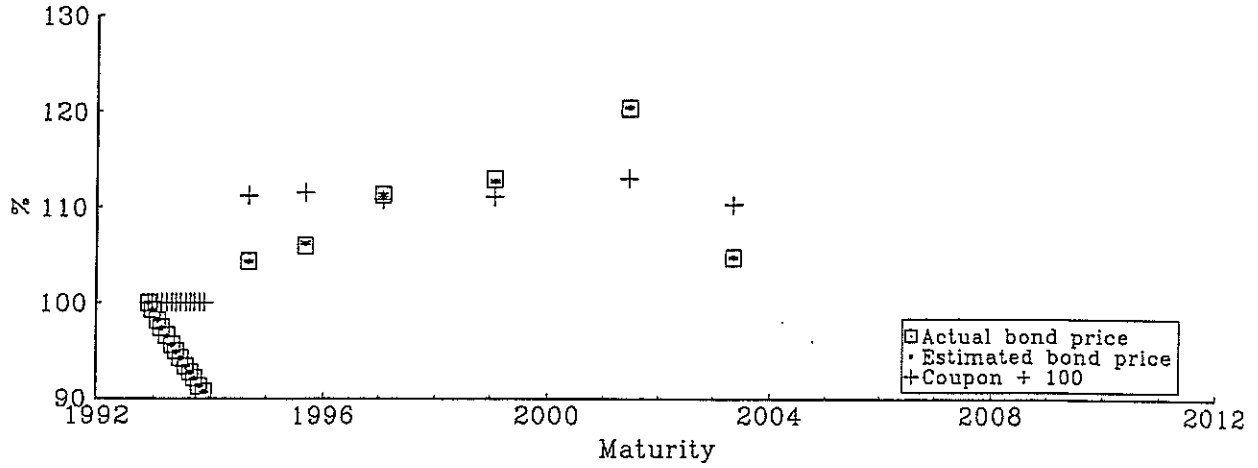


Figure 2b. NS2, Actual and Estimated Bond Prices, Nov 23, 1992

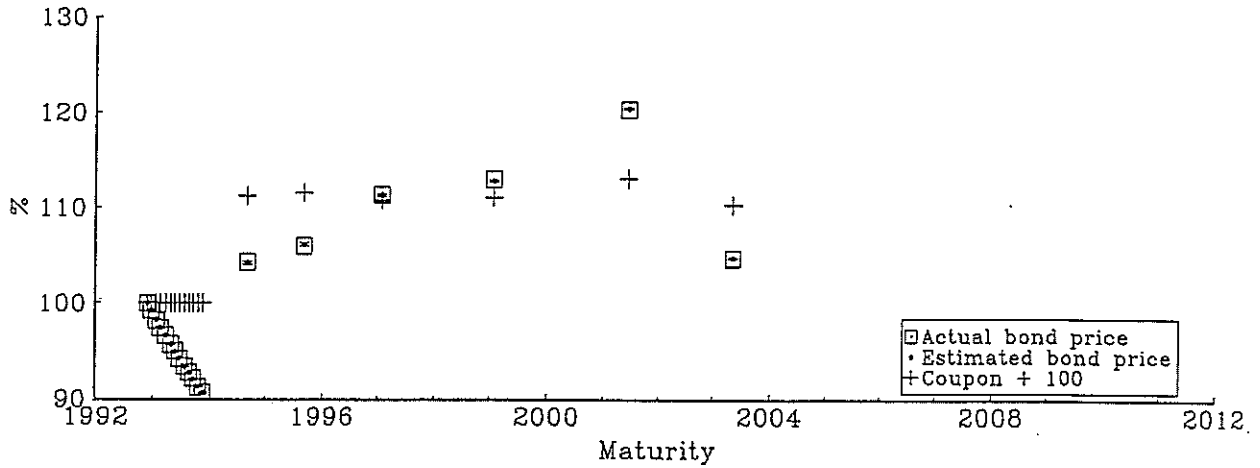


Figure 2c. LS, Actual and Estimated Bond Prices, Nov 23, 1992

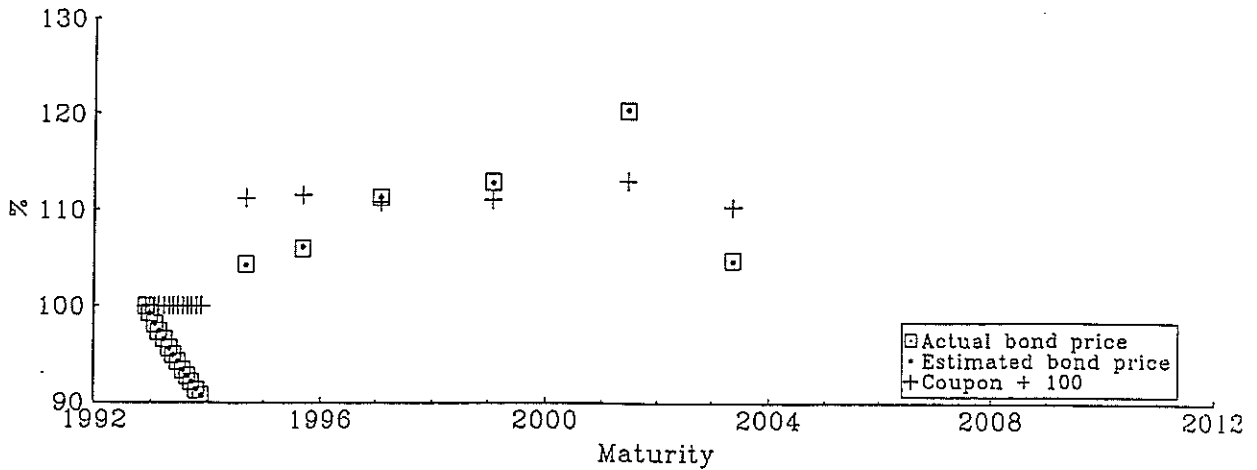


Figure 3a. NS1 (Nelson and Siegel, w/ restriction), April 16, 1993
95% confidence interval

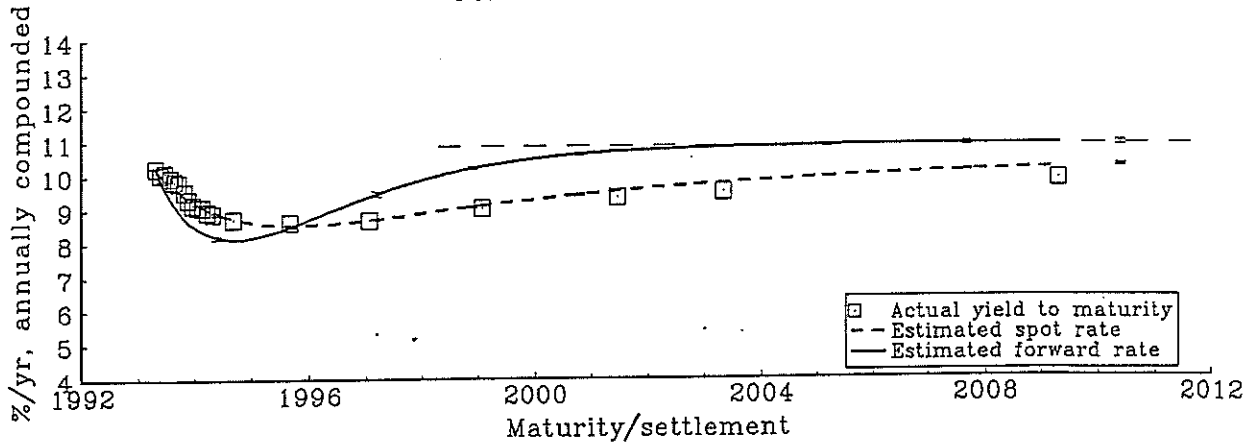


Figure 3b. NS2 (Nelson and Siegel, w/o restriction), April 16, 1993
95% confidence interval

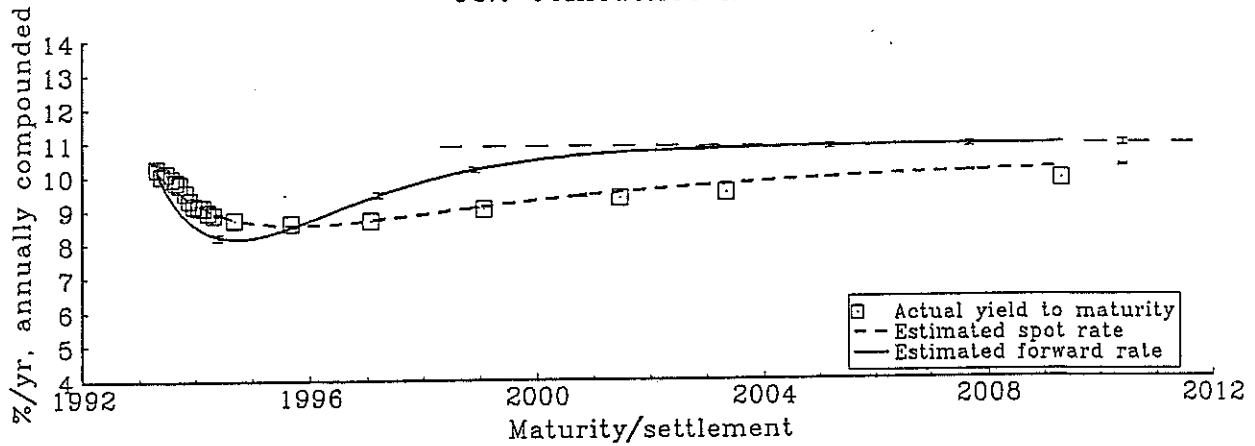


Figure 3c. LS (Longstaff and Schwartz, w/ restriction), April 16, 1993

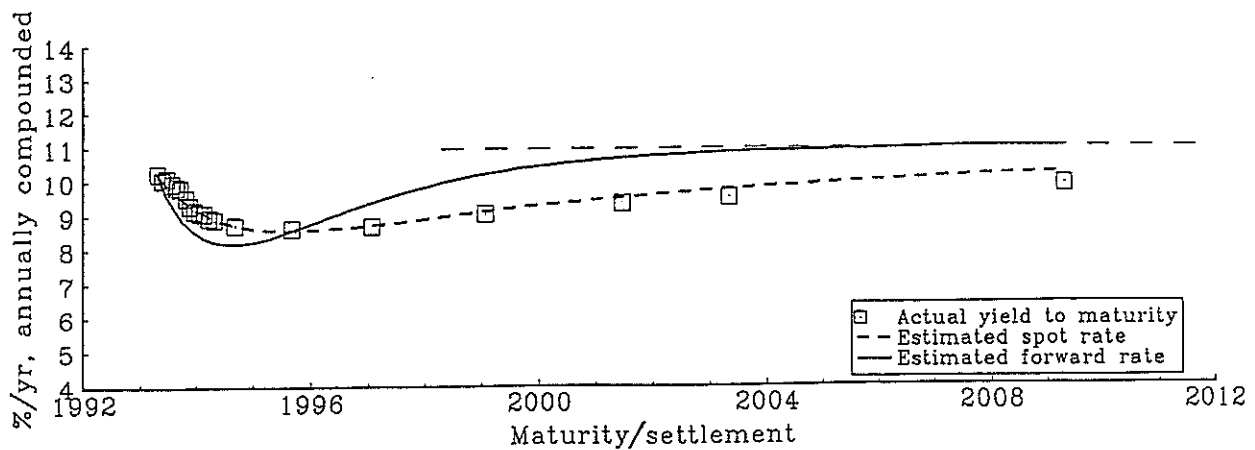


Figure 4a. NS1, Actual and Estimated Bond Prices, April 16, 1993

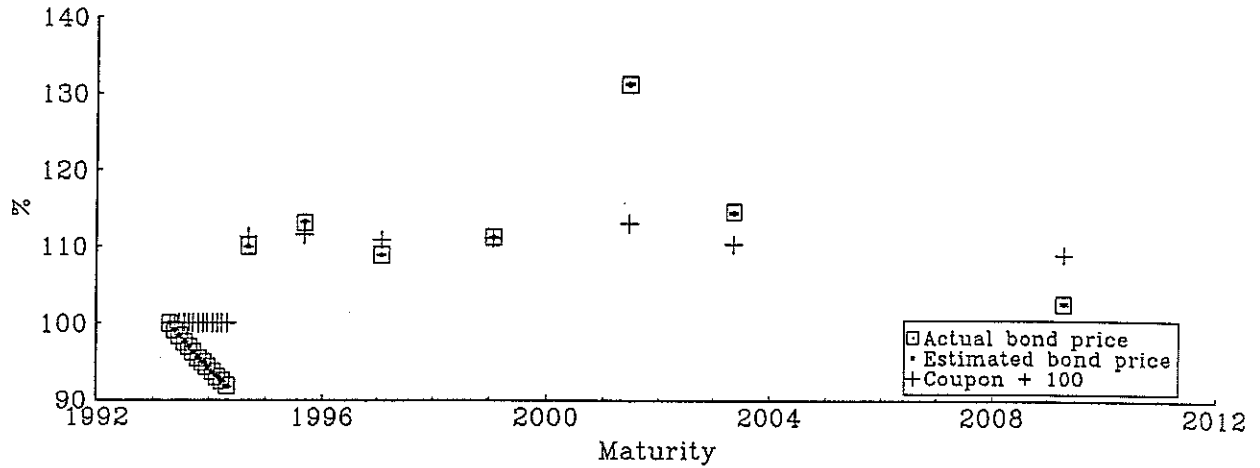


Figure 4b. NS2, Actual and Estimated Bond Prices, April 16, 1993

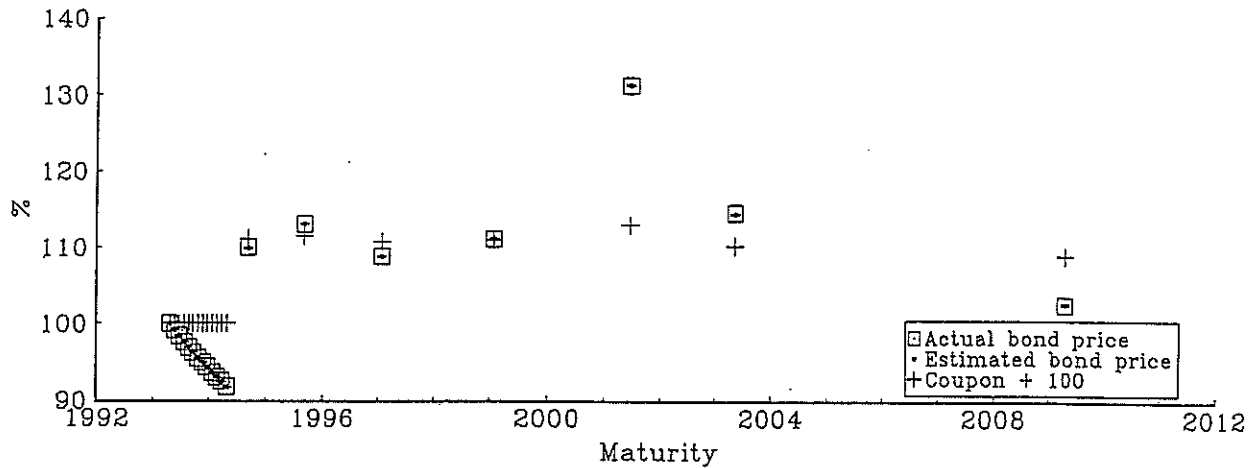


Figure 4c. LS, Actual and Estimated Bond Prices, April 16, 1993

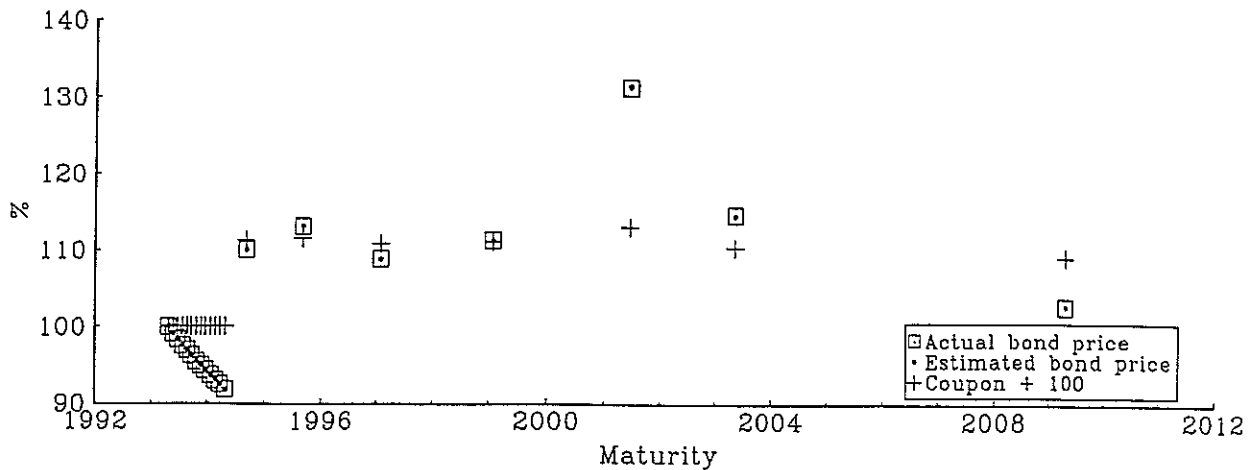


Figure 5a. NS1, Spot Rates

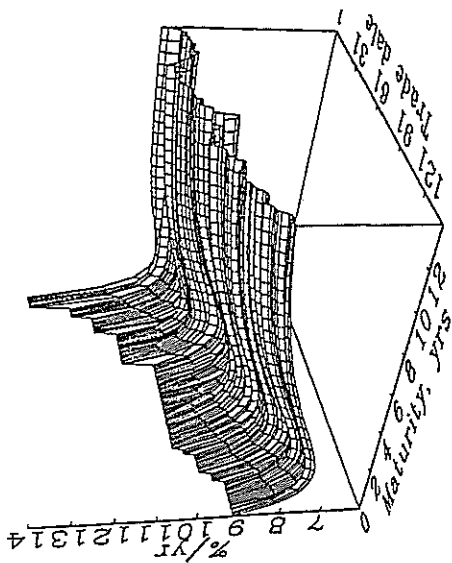


Figure 5b. NS2, Spot Rates

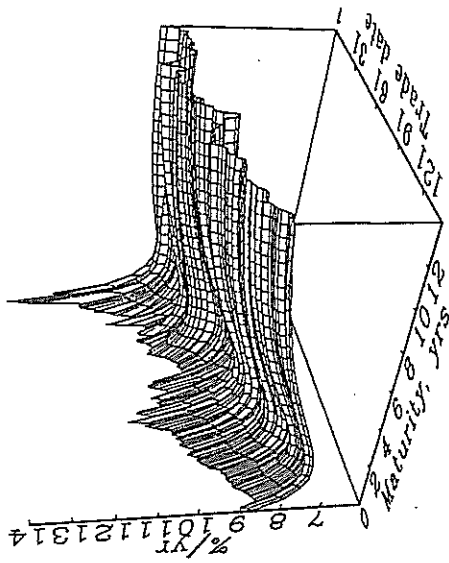


Figure 5c. LS, Spot Rates

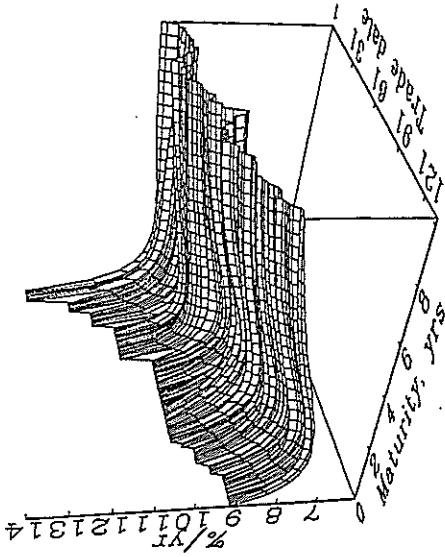


Figure 6a. NS1, Forward Rates

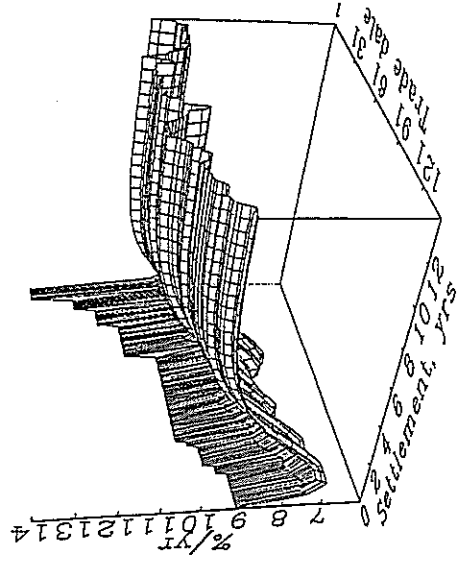


Figure 6b. NS2, Forward Rates

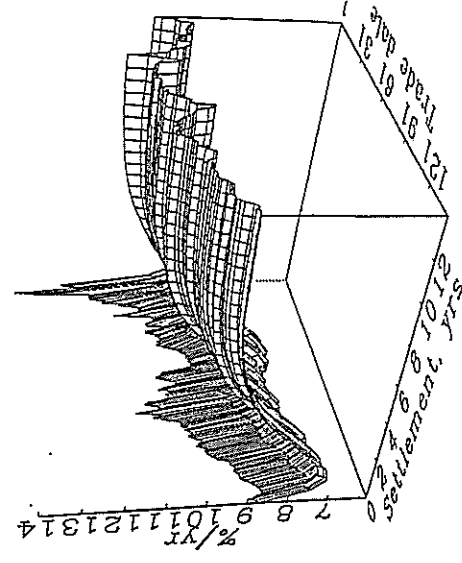
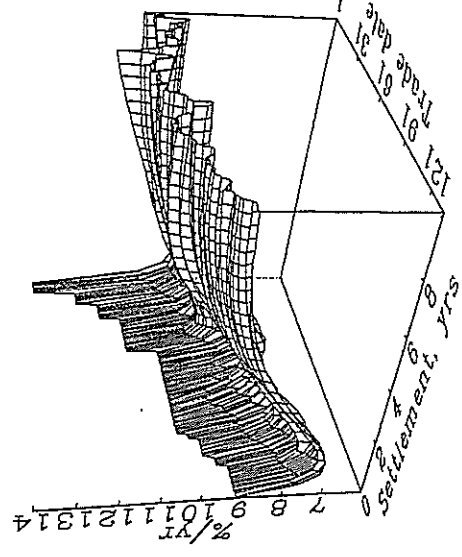


Figure 6c. LS, Forward Rates



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